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Einige Probleme der Konstruktion und des Baues von kernphysikalischen Beschleunigungsanlagen für einige MeV Teilchenenergie.

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INHALT. — 1. Allgemeine Betrachtungen. – 2. Die Beschreibung der ausgeführten Beschleunigungsanlagen des Zentralforschungsinstitutes der Ungarischen Akademie der Wissenschaften.

1. – Allgemeine Betrachtungen.

Da es heute schon Beschleunigungsanlagen gibt, welche schwere Teilchen auf einige 10^9 eV beschleunigen können, stellt sich die Frage, ob es überhaupt zweckmäßig ist, Apparate zur Erzeugung von Teilchen mit einer Energie von einigen 10^6 eV zu bauen. Im nachstehenden beabsichtige ich, als Einleitung, eben jene Gesichtspunkte anzuführen, die bei der Beurteilung der Anwendbarkeit der einzelnen Apparatetypen maßgebend sind.

Die Tatsache, daß die kosmischen Strahlen dem Forscher Teilchen mit einer Energie von sogar 10^{15} eV zur Verfügung stellen, beweist, daß die Endenergie der Teilchen — zwar ein äußerst wichtiges — doch keinesfalls das einzige Kennzeichen der Anwendbarkeit eines Apparates ist, denn aus diesem Gesichtspunkte könnte auch die Zweckmäßigkeit des Baues eines Kosmotrons in Frage gestellt werden. Diese Frage ist natürlich leicht zu beantworten:

die Intensität des Kosmotrons beträgt 10^9 Korpuskel/s, wogegen die Teilchenzahl der auf die gleiche Fläche entfallenden kosmischen Strahlen 10^2/s ist. Das zweite Argument ist, daß Energie und Qualität der künstlich beschleunigten Korpuskel leicht zu behandeln und zu regeln sind, können also für die uns eben interessierenden Ionensorten und Energiebereiche eingestellt werden.

Für die Beurteilung der Verwendbarkeit eines Apparates sind nun aus rein physikalischem Gesichtspunkte folgende Fragen entscheidend:

1) Ob in dem, mit dem Apparat erreichbaren Energiebereich Versuche von praktischer oder von theoretischer Bedeutung durchgeführt werden können.

2) Welcher Typ weist zwischen Geräten, die im gleichen Energiebereich arbeiten, die größte Intensität auf.

3) Bei welchem Apparat kann man die Energie am besten in der Hand halten, d.h. am genauesten einstellen und welcher bietet die größte Energiehomogenität.

Schließlich müssen natürlich auch Wirtschaftlichkeits- und wirtschaftspolitische Prinzipien in Betracht gezogen werden: mit welchem Aufwand, mit welchen in-, bezw. ausländischen Materialien, bezw. Hilfsapparaturen der Bau der fraglichen Anlagen verbunden ist.

Wie es aus der wissenschaftlichen Literatur hervorgeht, können mit Strahlen beschleunigter Elektronen oder schwerer Teilchen von 0.03 MeV aufwärts bis zu beliebigen Energiewerten Untersuchungen sowohl von praktischer, als auch von theoretischer Bedeutung vorgenommen werden.

Die Abbildung 1 enthält die interessanteren Probleme der physikalischen Forschung der letzten Jahre nebst Anführung des Energiebereiches.

Wie man sieht, bietet der Bereich der kleineren Energien auch eine vielfältige Forschungsmöglichkeit. Von den äußerst große Energie leistenden Einrichtungen sind natürlich Ergebnisse von viel größerer prinzipieller Wichtigkeit zu erwarten.

Der Abbildung 1. sind auch die Bereiche der Anwendbarkeit der einzelnen Gerätetypen zu entnehmen. Dieses Diagramm werden wir nun im nachfolgenden zur Bestimmung einer zweckmäßigen Einrichtung eines Atomforschungsinstitutes benutzen.

Bis auf Energien von einigen MeV kommen nur direkte Beschleuniger in Betracht, und zwar vier Typen derselben: der Freiluft- bezw. der unter Druck arbeitende Van de Graaff Generator und der Freiluft-, bezw. der unter Druck arbeitende Kaskadengenerator.

Die Van de Graaff Beschleunigertypen, bezw. deren Spannungsbereiche sind heute schon ziemlich genau definiert. Die Van de Graaff Freiluft-Spannungsquelle ist einfach, billig und ohne technische Schwierigkeiten herstellbar. Ihre Beschleunigungsanlage bietet natürlich ebensoviel Probleme, wie jede beliebige andere Beschleunigungsanlage. Als Spannungsquelle hat diese Ein-

richtung heutzutage hauptsächlich die Rolle, in Unterrichtsanstalten andere elektrostatische Maschinen zu beseitigen.

In Energiebereichen von 1 bis 5 MeV ist der unter Druck arbeitende Van de Graaff Generator zur Zeit einer der praktischesten Gerätetypen. Man kann seine Spannung besonders gut stabilisieren, so daß der Apparat hinsichtlich Regelbarkeit den Anforderungen am besten von allen ähnlichen Anlagen ent-

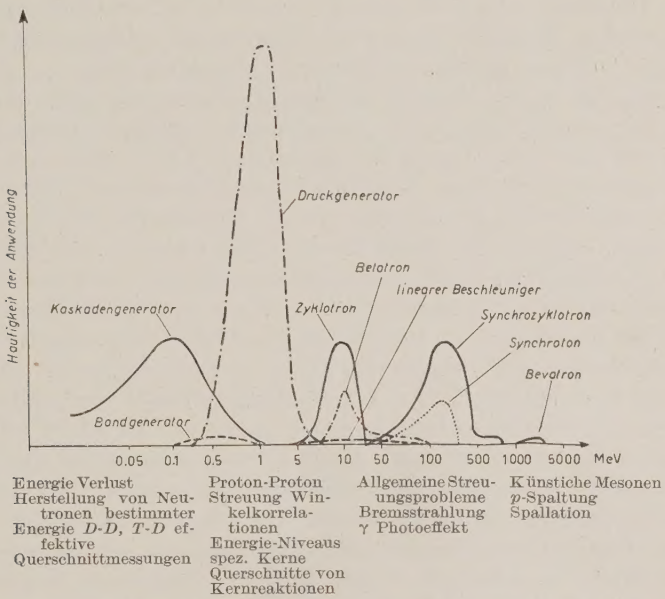

Abb. 1. – Die Verteilung der mittels Beschleunigungsanlagen durchgeführten verschiedenen Versuche als Funktion der Energie. Die Kurven erhielten wir auf Grund einer statistischen Auswertung von hauptsächlich in der Physical Review in den Jahren 1951-1954 erschienenen Artikel.

spricht. Seine Stromstärke ist auch hinreichend: einige hundert μA sind ohne Schwierigkeit zu erreichen. Einen Nachteil bildet nur die verwickelte technische Ausführung und der damit verbundene hohe Preis des Apparates. Sein rationeller Grenzspannungswert beträgt ungefär 6 MV.

Für den Freiluft-Kaskadengenerator ist als wichtigstes Kennzeichen die große Stromstärke zu betrachten, welche prinzipiell einige mA beträgt. Seine Konstruktion ist verhältnismäßig nicht zu kompliziert. Zufolge ihrer Betriebssicherheit ist diese Anlage unterhalb der Grenze von 1 MV eine der am meisten verbreiteten Typen. Sein rationeller Grenzspannungswert liegt ungefähr bei 1.5 MV, jedoch eher unterhalb, als oberhalb dieses Wertes. Wert und Leistungsfähigkeit dieser Anlage werden im Allgemeinen stark überschätzt.

Der Kaskadengenerator mit Pressgasisolierung stellt einen neuen Typus dar. Prinzipiell vermag er dem Van de Graaff Generator eine starke Konkurrenz zu machen, obzwar bei dieser Anlage das elektrische Feld bei weitem nicht so günstig ausgebildet werden kann. Somit ist der rationelle Grenzspannungswert bei dieser Anlage fast sicher niedriger, als jener der Van de Graaff Generatoren unter Druck.

Bei Belastung zeigen die Kaskadengeneratoren bedeutende Spannungsschwankungen. Prinzipiell kann zwar dieser Nachteil eliminiert werden, über praktische Angaben verfügen wir jedoch diesbezüglich nicht.

Da sich die in der Literatur enthaltenen Gerätebeschreibungen im allgemeinen nur auf die Angabe der Betriebsdaten (manchmal bloß auf jene der Höchstwerte der Spannungsquelle) beschränken (z.B. 1 MV, 4 mA), läßt sich die Leistungsfähigkeit dieser Anlagen nicht rasch beurteilen. Mit der Angabe der Spannung der Spannungsquelle und der Stromstärke der Ionenquelle ist schon mehr besagt. Für physikalische Untersuchungen gilt natürlich als einziger Kennwert *die Angabe der zusammengehörenden Werte der Beschleunigungsspannung und der Targetstromstärke*. In solchen Fällen stellt sich nun oft heraus, daß die im Titel der Beschreibung angeführte Zahl vielmehr als *Name* und nicht als technische Angabe des Apparates zu betrachten ist. Unser Van de Graaff Freiluftgenerator mit den technischen Daten von 1 MV - 180 μA ergab z.B. bei einer Höchstspannung von 1.2 MV nur 0.8 MV, wenn auch das Beschleunigungsrohr angekoppelt war; bei Beschleunigung fiel die Spannung auf 0.6 zurück: nebst einer betriebsmäßigen Spannung von 0.3-0.4 MV war ein Targetstrom von 10 μA zu messen. Obzwar diese Daten sich natürlich nur bei vorhergehenden Untersuchungen ergaben und sehr schlechte, korrigierbare Werte darstellen, die in der Tat auch ausgebessert wurden, zeigen sie klar, daß die wahre Beschleunigungsspannung — abgesehen von speziellen Einzelfällen — wesentlich niedriger ist, als die Betriebsspannung des Generators.

Wie bereits erwähnt, ist für den Physiker nur der bei einer gegebenen Spannung erzeugte Targetstrom und nicht die Spannung der Spannungsquelle und der ihr zu entnehmende maximale Strom kennzeichnend. Die Dimensionierung der Spannungsquellen ist nämlich eine verhältnismäßig einfachere elektrotechnische Aufgabe.

Die Höchstspannung der ganzen Beschleunigungsanlage hängt vom Vakuumdurchschlag ab und der Targetstrom ist nicht durch den, der Ionenquelle entnehmbaren Strom, sondern durch die Fokussierungsschwierigkeiten bestimmt. Diese Schwierigkeiten nehmen bei wachsender Spannung infolge der Zunahme der Länge des Beschleunigungsrohres und bei wachsender Stromstärke wegen Ansteigen der Raumladung in gesteigertem Maße zu. Als Neutronenquelle ist also der Bau von Geräten mit verhältnismäßig niedriger Spannung, jedoch großer Stromstärke zweckmäßig.

Auf Grund des Vorangeführten kann folgendes festgelegt werden: Der Kas-

kadengenerator mit einer Spannung von einigen Hunderttausend Volt erweist sich als eine vielfältig verwendbare und dabei billigste Neutronenquelle. Wird dieser Apparat mit einem Tankgenerator von 2-4 MV ergänzt, so ist man imstande, wie dies aus der Abbildung 1 ersichtlich ist, sich an den Großteil der kernphysikalischen Forschungen der ganzen Welt anzuschließen. Die Zweckmäßigkeit eines einzigen Kaskadenbeschleunigers von ungefähr 1 MV ist äußerst fraglich.

Die Abbildung 2 gibt über die Preise Aufschluß: hieraus folgt, daß die Beschleunigung der Elektronen unvergleichlich einfacher ist. Ungefähr mit dem gleichen Aufwand an materiellen Mitteln und an geistiger Energie können fast um zwei Größenordnungen höhere Elektronenergien erzielt werden.

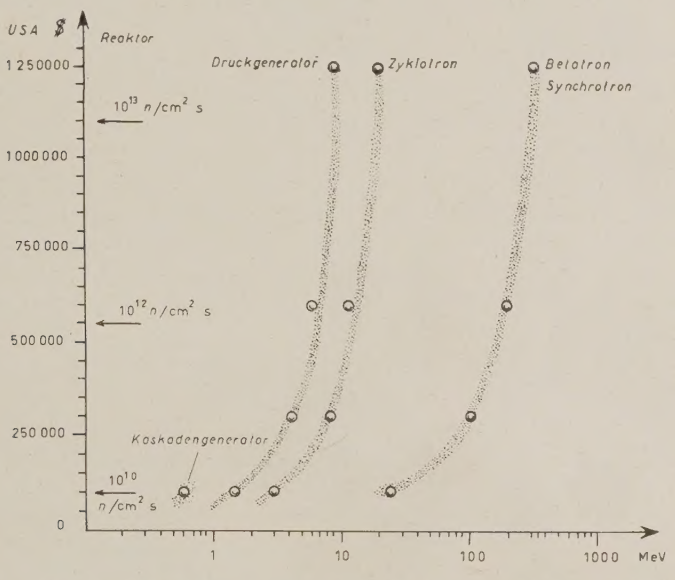

Abb. 2. – Die Herstellungskosten verschiedener Generatortypen als Funktion der erwarteten Energie. An der linken Seite der Abbildung sind mit Pfeilen die Kosten von Reaktoren mit verschiedenem Neutronenflusse ersichtbar. Laut *Nucleonics* (April 1954).

Es muß wieder betont werden, daß ein Kaskadengenerator von 500 keV Teilchenenergie als Neutronenquelle oder ein Tankgenerator von 5 MeV, durch einen Reactor resp. Zyklotron von 20, oder sogar Synchrozyklotron von 200 MeV nicht *ersetzt*, sondern nur *ergänzt* werden kann.

2. – Die Beschreibung der ausgeführten Beschleunigungsanlagen des Zentralforschungsinstitutes der Ungarischen Akademie der Wissenschaften.

An der Abteilung für Atomphysik des Zentralforschungsinstitutes für Physik wurden anfangs die oben beschriebenen Prinzipien nicht befolgt: wir trachten jetzt nachträglich, uns diesen soweit wie möglich anzupassen. Nach Anfangsuntersuchungen mit Freiluft-Bandgeneratoren (Abbildung 3) haben wir an der Abteilung für Atomphysik vorerst den Bau eines 800 kV Kaskadengenerators und nachher den Bau eines 4 MV Tankgenerators begonnen. Erst jetzt kehren wir, neben den oben erwähnten beiden Beschleunigungsanlagen, auch auf den ganz modernen Kaskadenbeschleuniger von 600 kV zurück.

Alle diese Geräte wurden durch die Abteilung für Atomphysik selbst konstruiert und ausgeführt, abgesehen von Arbeitsstücken ganz großer Dimensionen. Die Dimensionierung haben wir natürlich auf Grund der in der Literatur bekannten Konstruktionen durchgeführt. (Siehe Literaturangabe).

Eine Stufe des Kaskadengenerators (Abbildung 4) besteht aus zwei Kondensatoren älteren Fabrikates Meirowsky mit einer Kapazität von 0.01 μF und aus zwei Siemens-Vakuumventilröhren mit einer Heizleistung von 120 W und einer Sperrspannung von 230 kV. Die Kapazität des an den Transformator angeschlossenen Kondensators beträgt 0.02 μF. Die Höchstspannung einer einzigen Stufe hat den Wert von 200 kV. Die Primärspannung des Transformators wird von einem Umformer geliefert, welcher durch Einstellen der Erregerspannung zwischen 0 und 300 V regulierbar ist. Zwecks Minderung des Spannungsabfalls ist die Frequenz für $f = 500\ s^{-1}$ gewählt. Der Maximalwert der Sekundärspannung des Transformators ist 100 kV. Bei diesen Daten beträgt der Spannungsabfall 8400 V/mA und die Welligkeit 2000 V/mA.

Die in der Literatur beschriebenen Kaskadengeneratoren unterscheiden sich in erster Linie in der Ausbildung des Hochspannungsfeldes, ferner in der Art der Röhrenheizung von einander. Heute wird fast ausschließlich das durch die Philips-Werke eingeführte Modell mit oberer Schutzelektrode der separaten Endelektrode bevorzugt. Diese letztere Lösung bietet den Vorteil einer kleineren Bauhöhe, die vorher erwähnte sichert jedoch eine größere Homogenität des Feldes. Bei niedrigeren Spannungen erweist sich die letztere Lösung (Abbildung 9) als vorteilhafter. Die Heizung der Ventilröhren erfolgt entweder durch isoliert angetriebene Dynamos, oder mittels der auf den Kondensatorsäulen hinaufgeführten Hochfrequenz. Dieses Problem wird bei der neuesten Lösung, durch die Anwendung von Trockengleichrichtern, vollkommen beseitigt.

Vom technischen Gesichtspunkte aus kann die Dynamoheizung am einfachsten verwirklicht werden.

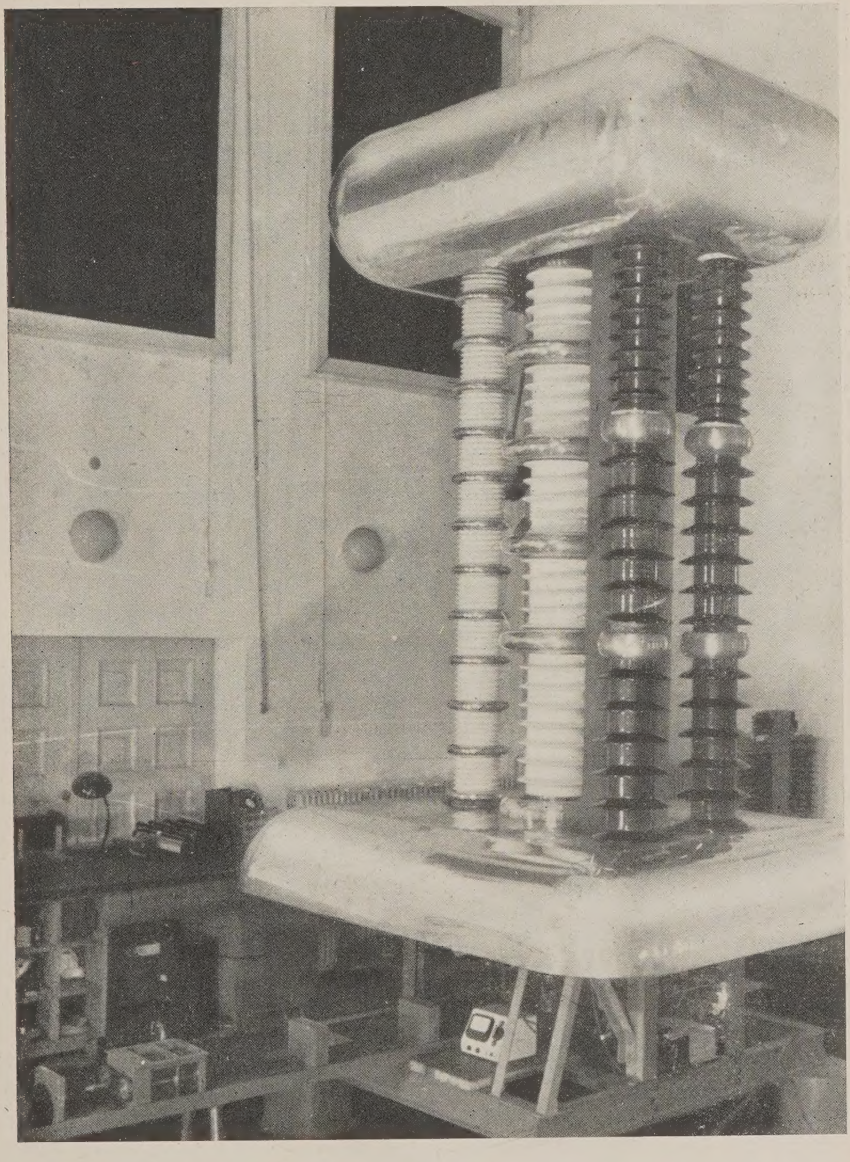

Abb. 3. – Freiluft Bandgenerator für 200-800 keV Elektronenbeschleunigung.

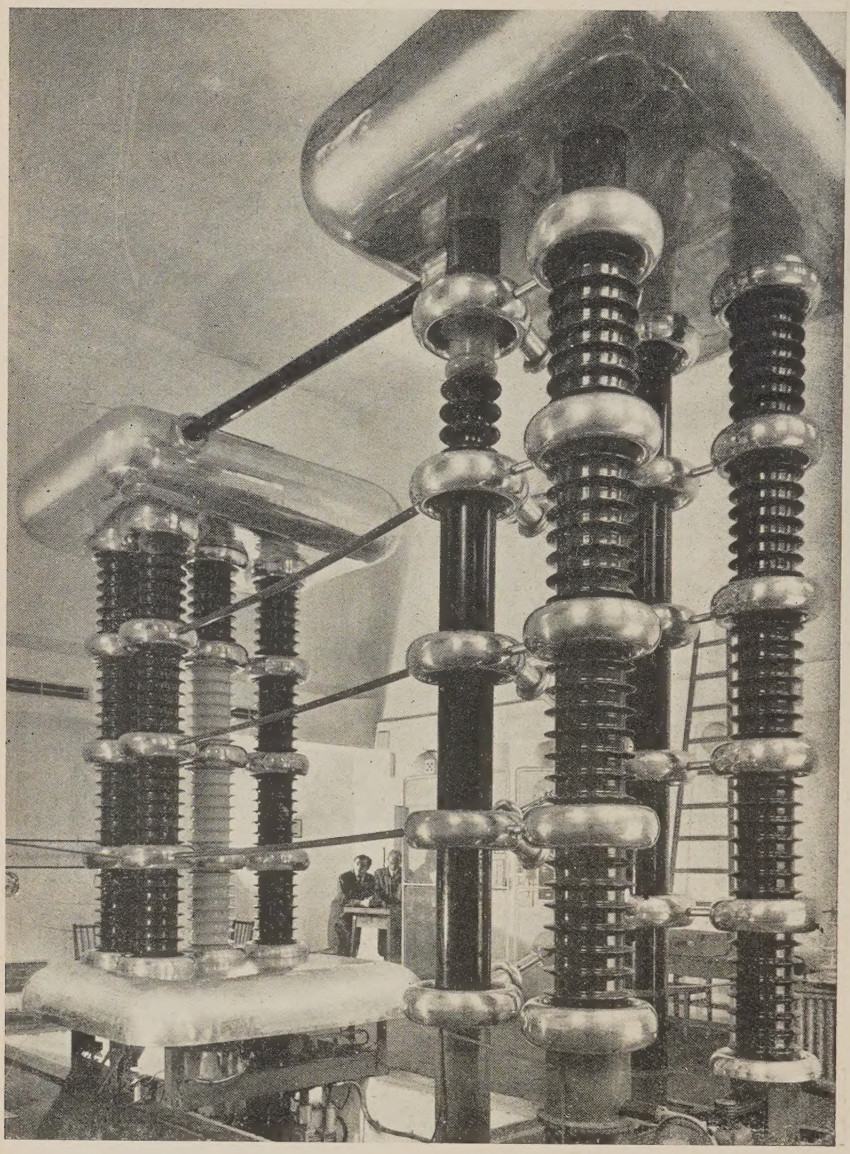

Abb. 4. – Lichtbild der 800 keV Beschleunigungsanlage, welche von der Atomphysikalischen Abteilung des Zentralforschungsinstitutes für Physik in Budapest verfertigt wurde.

Die von uns gewählte Konstruktion (Siehe Abb. 5) entspricht den Anforderungen großer mechanischer Stabilität, verhältnismäßiger Geräuschlosigkeit und separater Kühlungsmöglichkeit, ihre Montierbarkeit ist jedoch nicht als glücklich zu bezeichnen. Die Reguliervorrichtungen der Dynamos sind zwar

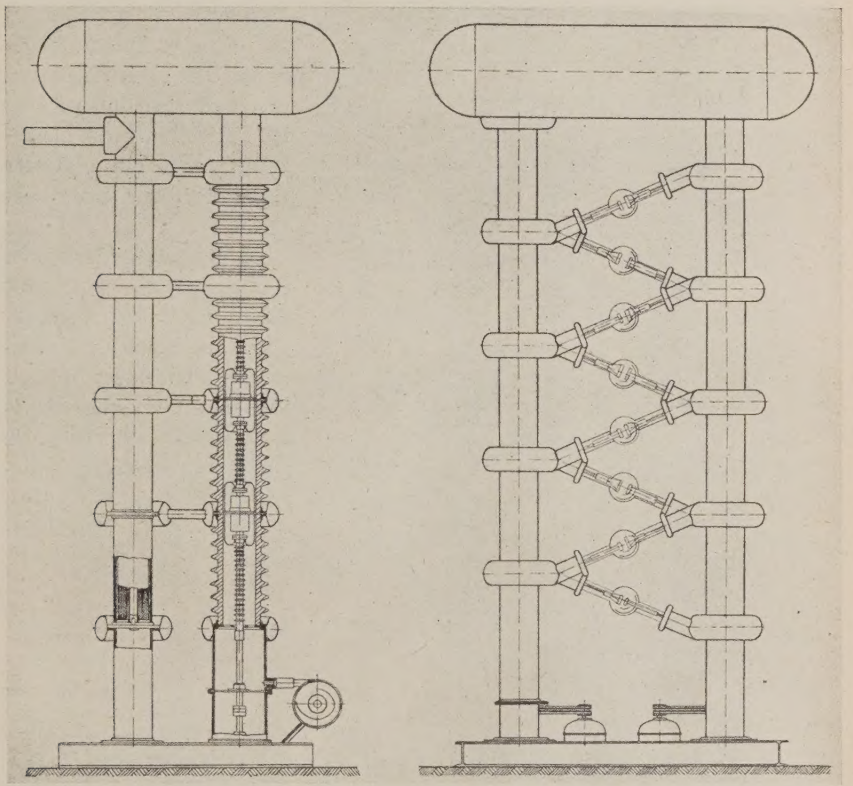

Abb. 5. – Abbildung des Spannungsteiles der Beschleunigungsanlage: der 800 keV Kaskadengenerator der Atomphysikalischen Abteilung.

auch ohne Abnehmen der Abrundungen zugänglich, Reparatur und Reinigung der Dynamos sind jedoch nur nach vollkommener Zerlegung der Säulen möglich. Mit Rücksicht auf die für die Dynamos garantierte lange Lebensdauer scheint dies nicht gefährlich zu sein, da die Anlage sowieso jährlich plangemäß gereinigt und zerlegt wird. Bis heute hat sich dies (bei einer Betriebszeit von ungefähr anderthalb Jahren) als genügend erwiesen. Unsere Abteilung hat neuerlich auch den Bau eines 600 kV Kaskadengenerators mit Hochfrequenzheizung beendigt, wir wollen jedoch über die hiemit erworbenen Erfahrungen anderswo berichten.

Die Berechnung des elektrischen Feldes ist nur mit einer ziemlich groben, von verschiedenen Seiten aus erfolgten Näherung möglich. Die auf diese Weise erhaltenen Dimensionen wurden durch Aufbau einer Stufe experimentell kontrolliert.

Die Einzelteile der Achsenverbindung sind in Abb. 5 ersichtlich. Der Antriebsmotor ist außerhalb der Säule angebracht, nachdem er innerhalb der Säule bei Durchschlägen oft in Kurzschluß geraten ist. Hiemit wurde gleichzeitig auch ein glatterer Gang der Einrichtung erreicht.

Die Spannung der ersten Generatorstufe wird mit einem in Ölbad befindlichen, aus Ingelen-Fabrikaten konstruierten, bis zu 250 kV anwendbaren Meßwiderstand von 1640 MΩ gemessen. Die Eichung dieses Widerstandes haben wir mit einer Kugelfun-

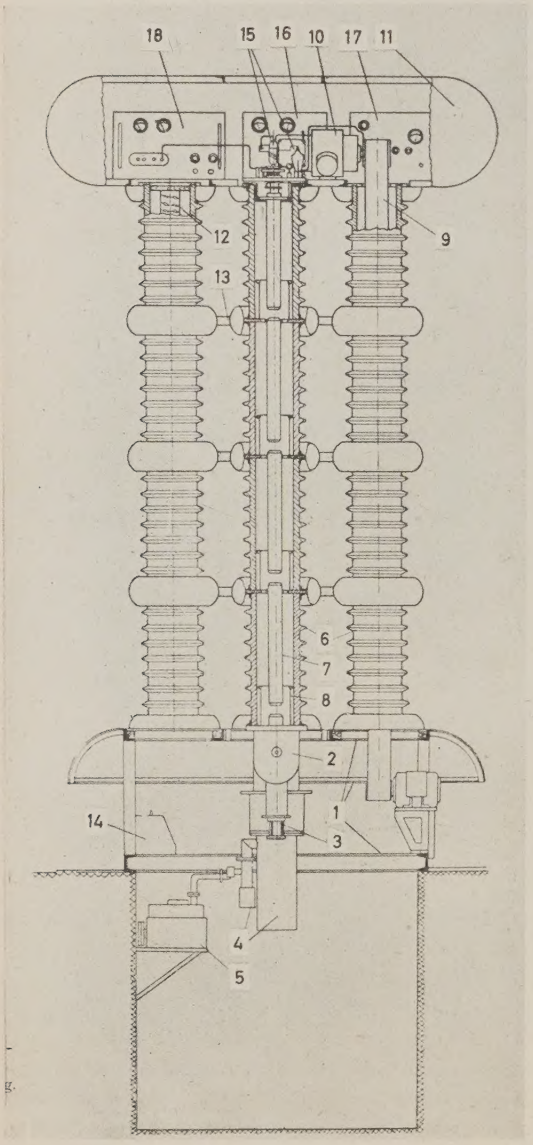

Abb. 6. – Aufbau des Beschleunigungsteiles. 1. Trägerkonstruktion. 2. Absaugerohr. 3. Auffänger des Beschleunigungsrohres. 4. Diffusionspumpe. 5. Vorvakuumpumpe. 6. Porzellanisolatoren. 7. Beschleunigungselektroden. 8. Antreibeband des 3×380 V Generators. 10. 3×380 V Generator. 11. Obere Elektrode. 12. Ableitwiderstand in Öl. 13. Elektrische Verbindungen. 14. Vakuummeter. 15. Hochfrequenz Ionenquelle und Oszillator. 16. Spannungsquelle der Absaugespannung der Ionenquelle. 17. Spannungsquelle der Vorfokussierung. 18. Spannungsquelle des Oszillators.

kenstrecke von 15 cm bezw. 75 cm Durchmesser, unmittelbar im Bereiche der Meßspannung vorgenommen, nachdem die einzelnen Teile des Widerstandes bereits bei niedrigeren Spannungen geeicht worden sind. Zur Kontrolle dient Spannung und Erregerstrom der Primärseite und außerdem ein Rotationsvoltmeter, der die Spannung des ganzen Generators mißt und bei niedrigeren Spannungen unmittelbar mit dem Widerstandsspannungsmesser verglichen worden ist.

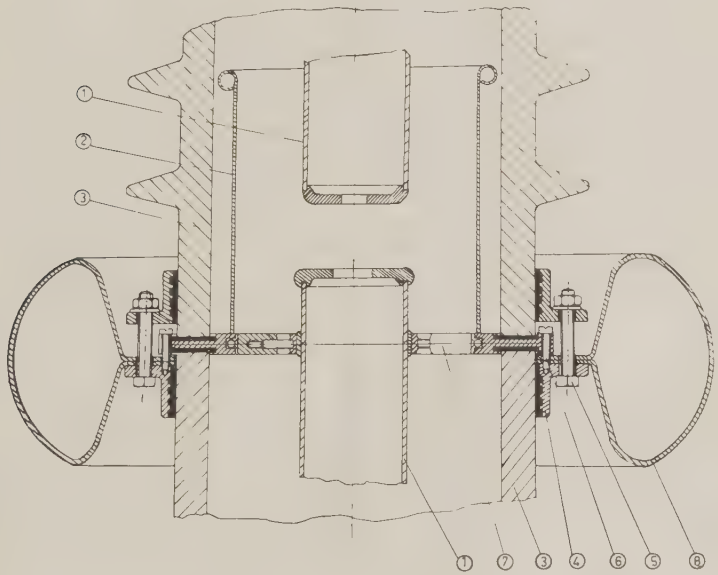

Abb. 7. – Das elektrische Linsensystem der Beschleunigung und Fokussierung. - 1. Beschleunigungselektroden. 2. Abschirmungselektrode. 3. Porzellanisolator. 4. Ringförmiges Zwischenglied. 5. Verbindungsschraube. 6. Stellschrauben des Zwischengliedes. 7. Kugelgelenk-Stellschraube der Elektroden. 8. Abschirmungskörper.

Das Beschleunigungsrohr befindet sich laut Anordnung Abb. 6 zwischen den drei Tragsäulen der oberen Elektrode. Ein Widerstand, der ursprünglich für Meßzwecke bestimmt und sich in einer der Tragsäulen befindet, sichert den Erdanschluß der oberen Elektrode. In der zweiten Tragsäule sind die zur Regulierung der Ionenquelle dienenden Kunststoffdrähte angebracht, während die dritte Säule das Antriebsband des zur Energieversorgung der Ionenquelle dienenden Generators enthält. Dieses Isolierband zeigt eine merkliche Tendenz, als ein selbsterregender Van de Graaff Generator zu arbeiten. Hiedurch kann die obere Elektrode selbst im Falle der Ausschaltung der Speise-

spannung des Kaskadengenerators einen beträchtlichen Spannungswert erreichen welcher manchmal auch einige hundert kV betragen kann; dies vermag, mit Rücksicht auf die Kondensatoren, auch ernste Betriebsunfälle zu verursachen, insofern die nötigen Vorsichtsmaßregeln nicht rechtzeitig getroffen werden. Das Evakuieren des Beschleunigungsrohres erfolgt mittels einer Öldiffusionspumpe mit Sauggeschwindigkeit von 1500 l/s, jedoch ohne jede Kühlfalle.

Die Fokussierungsprobleme ließen sich nur mit der sorgfältigsten Montierung lösen. Nach Abb. 7 hat es sich als zweckmäßig erwiesen, das Tragglied der Elektrode gesondert zu befestigen. Ebenso wichtig ist, daß die radiale Verschiebbarkeit und die Kippbarkeit der einzelnen Elektroden gesichert werden.

Als Ionenquelle haben wir wegen geringer Ionenstreuung und großem Atomionengehaltes die Hochfrequenz-Ionenquelle gewählt. (Abb. 8).

Der Maximalwert des erhaltenen Targetstroms betrug 500 μA; bei einer Spannung von 600 kV kann betriebsmäßig 300-400 μA erreicht werden.

In Abb. 9 führen wir auch den Hochfrequenz-Kaskadengenerator vor, ohne denselben hierorts eingehend zu behandeln.

Vor Konstruierung einer jeden größeren Anlage hat unsere Abteilung eingehende vorherige Untersuchungen durchgeführt, um die optimalen Verhältnisse der Ionenquelle zu bestimmen, um

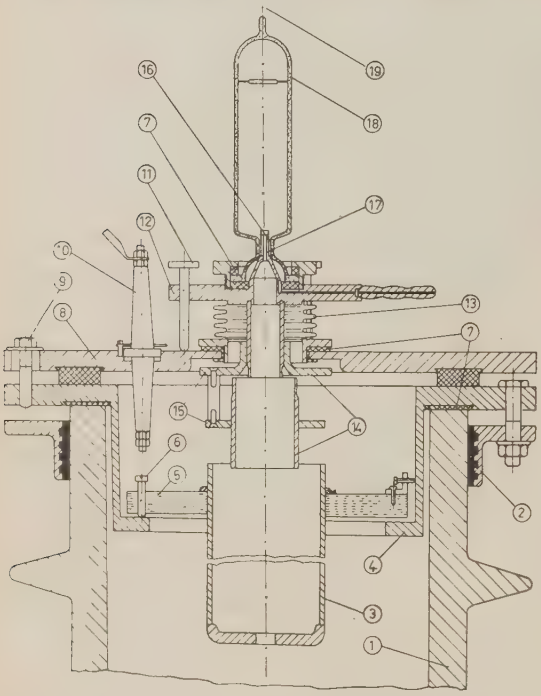

Abb. 8. – Anriß der Ionenquelle und des Vorfokussierungssystemes. - 1. Porzellanisolator. 2. Zwischenring. 3. Erste Beschleunigungselektrode. 4. Montierungseinsatz. 5. Befestigungsplatte der Elektroden aus Plexiglas. 6. Stellschrauben der Elektroden. 7. Vakuumdichtungen. 8. Abschlußplatte des Beschleunigungsrohres. 9. Schrauben der Abschlußplatte. 10. Hochspannungsdurchführungen. 11. Stellschrauben der Ionenquelle. 12. Unterlagplatte der Ionenquelle. 13. Tombak Federkörper. 14. Zylinder der Vorfokussierungslinse. 15. Montierungsplatte der Zylinder. 16. Absaugeröhrchen der Ionenquelle. 17. Quarzrörchen. 18. Ionenquelle. 19. Wolframelektrode.

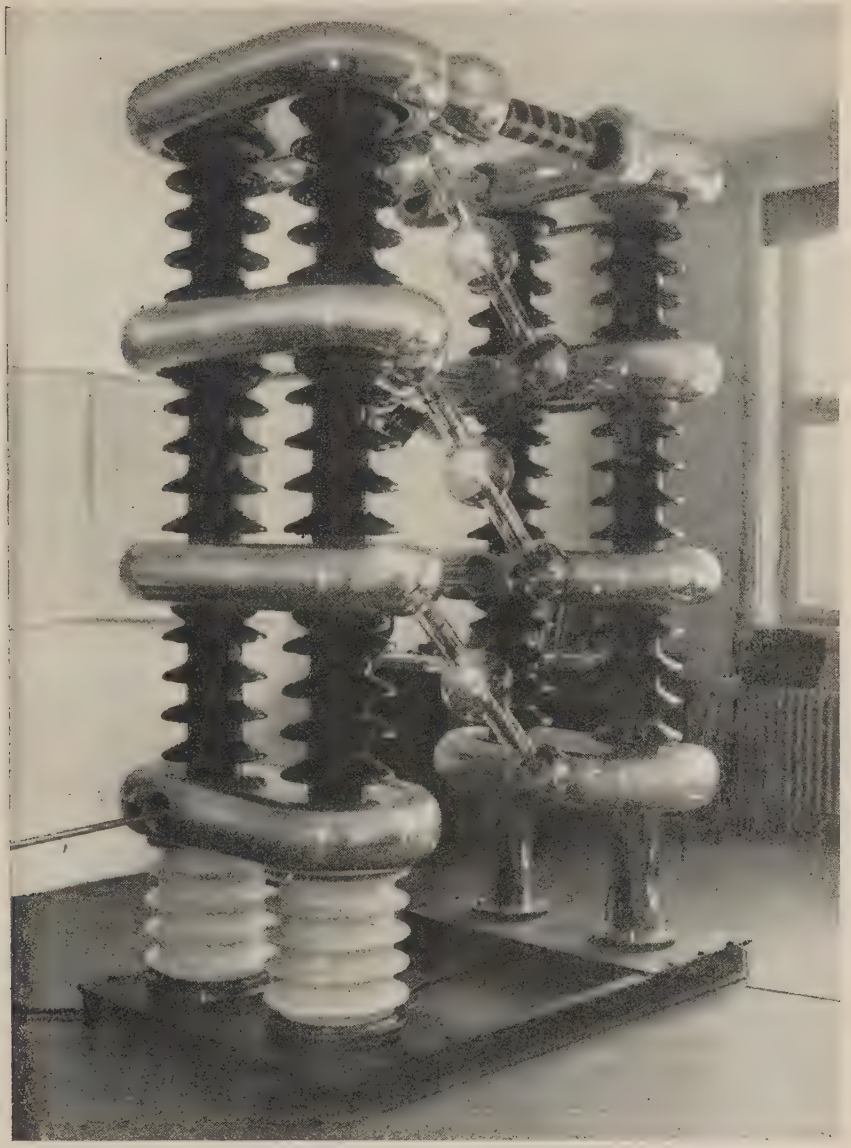

Abb. 9. – Der Kaskadengenerator mit Hochfrequenzheizung, welcher die Hochspannungsquelle der auf der Atomphysikalischen Abteilung im Bau befindlichen 600 keV Beschleunigungsanlage bilden. Die Hochspannungselektroden haben noch nicht ihre endgültige Form.

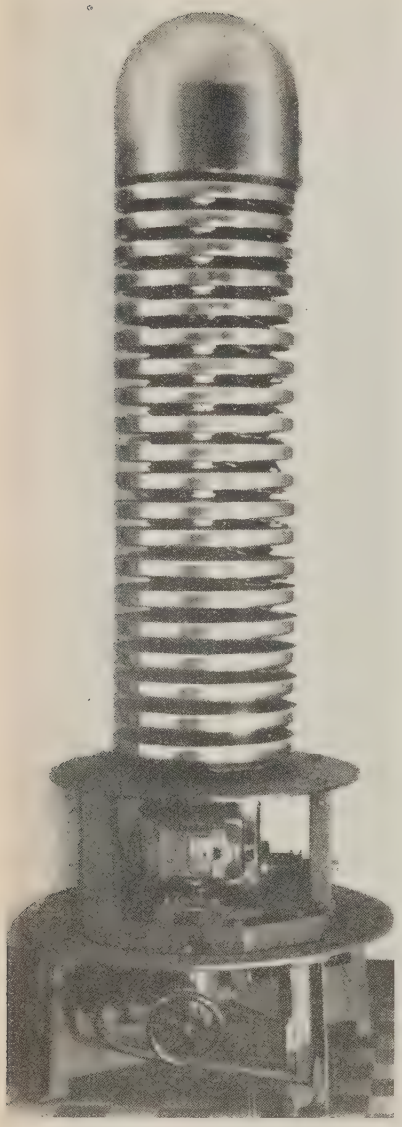

Abb. 10.
Tankgenerator-Modell für 1.7 MeV.

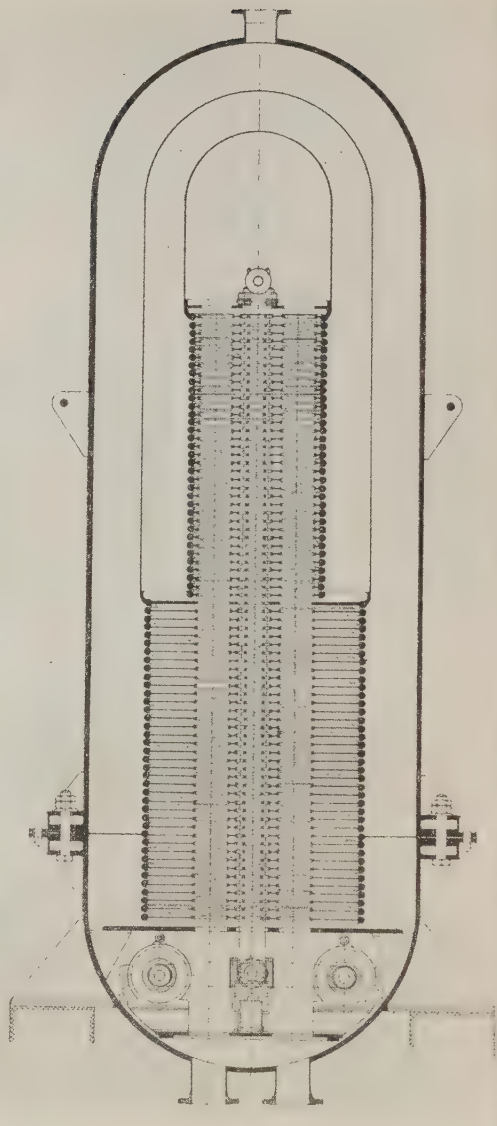

Abb. 11. – Anriß des 4 Millionen Volt Van de Graaff Druckgenerators.

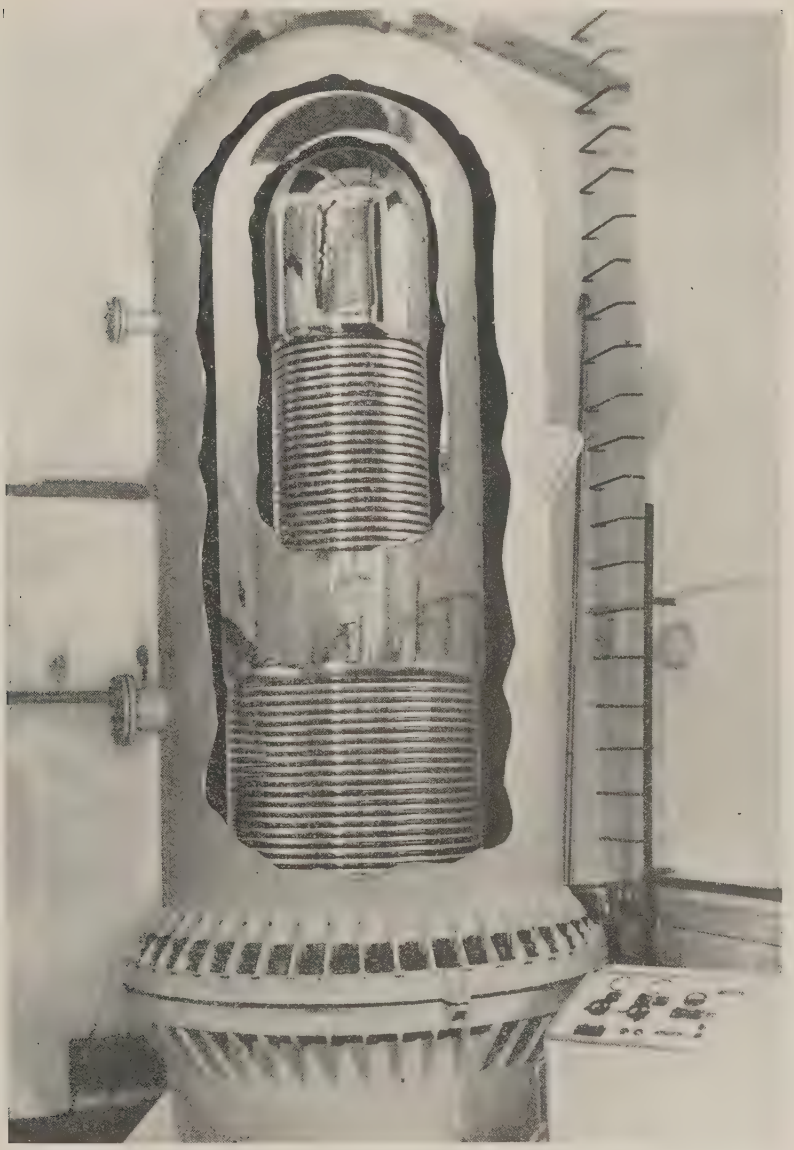

Abb. 12. – Lichtbild des 4 Millionen Volt Van de Graaff Druckgenerators.

den Durchschlagsmechanismus im Vakuum zu klären u.s.w.. Gleichfalls hat unsere Abteilung einen Tankgenerator von 1.5 MV gebaut, welcher nachher als Modell für den Bau eines großen Generators gedient hat. Mit diesem Apparat haben wir in Stickstoff unter 7 kg/cm² Druck eine Höchstspannung von 1.7 MV erreicht. (Abb. 10).

Die Konstruktion des großen Tankgenerators ist in Abb. 11 sein Lichtbild in Abb. 12 angeführt. Dieser Generator wurde im wesentlichen, abgesehen von der Ausbildung des unteren Teiles, den nach Literaturangaben bekannten Konstruktionen gemäß gebaut — hier möchte ich nur auf eine Einzelheit eingehen. Zwecks Homogenisierung des Feldes zwischen den Elektroden und der Wand des Behälters pflegt man Zwischenelektroden anzubringen und es erscheinen auch heute noch Artikel, die sich mit der Frage befassen, wie und mit wieviel Zwischenelektroden die Homogenität des Feldes am besten erzielt werden kann. Meines Erachtens ist diese Richtung sogar aus zwei Gründen verfehlt: einerseits ist es sogar von Vorteil, daß an der Wand des Behälters eine verhältnismäßig kleine Feldstärke herrscht, gegenüber der Feldstärke an der inneren Elektrode, deren vollkommene Glattheit technologisch unvergleichbar leichter zu sichern ist, als die der Behälterwand; andererseits ist die Spannung nicht durch den Durchschlag zwischen den Elektroden, sondern durch den Überschlag neben dem Bande oder den Tragsäulen begrenzt. Wenn der Durchschlag zwischen den Elektroden entsteht, kann die Durchschlagsfestigkeit des Füllgases durch sinngemäße Erhöhung des Druckes soweit gesteigert werden, daß die Hochspannung endlich durch den Kriechweg bestimmt wird. Als Bestes erscheint mir der mit einer *einzigen* Elektrode und mit einem Betriebsdruck von 15-20 kg/cm² arbeitende und mit $N_2 + SF_6$ gefüllte Generator.

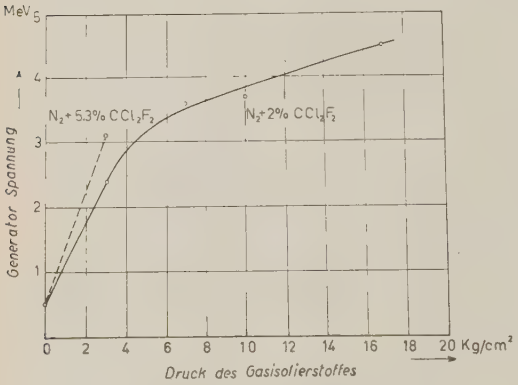

Abb. 13. – Zusammenhang zwischen der Maximalspannung, welche mit dem Druckgenerator erreicht werden kann, und dem Druck des Gasisoliermittels.

Laut unserer Messung war bei unserem Generator die erreichte Höchstspannung 4.5 MV (Abb. 13).

Außerdem leistet die Abteilung Hilfe zur Bildung der zukünftigen Forscher durch die Bedienung des Freiluft-Beschleunigungsapparates für Elektronen

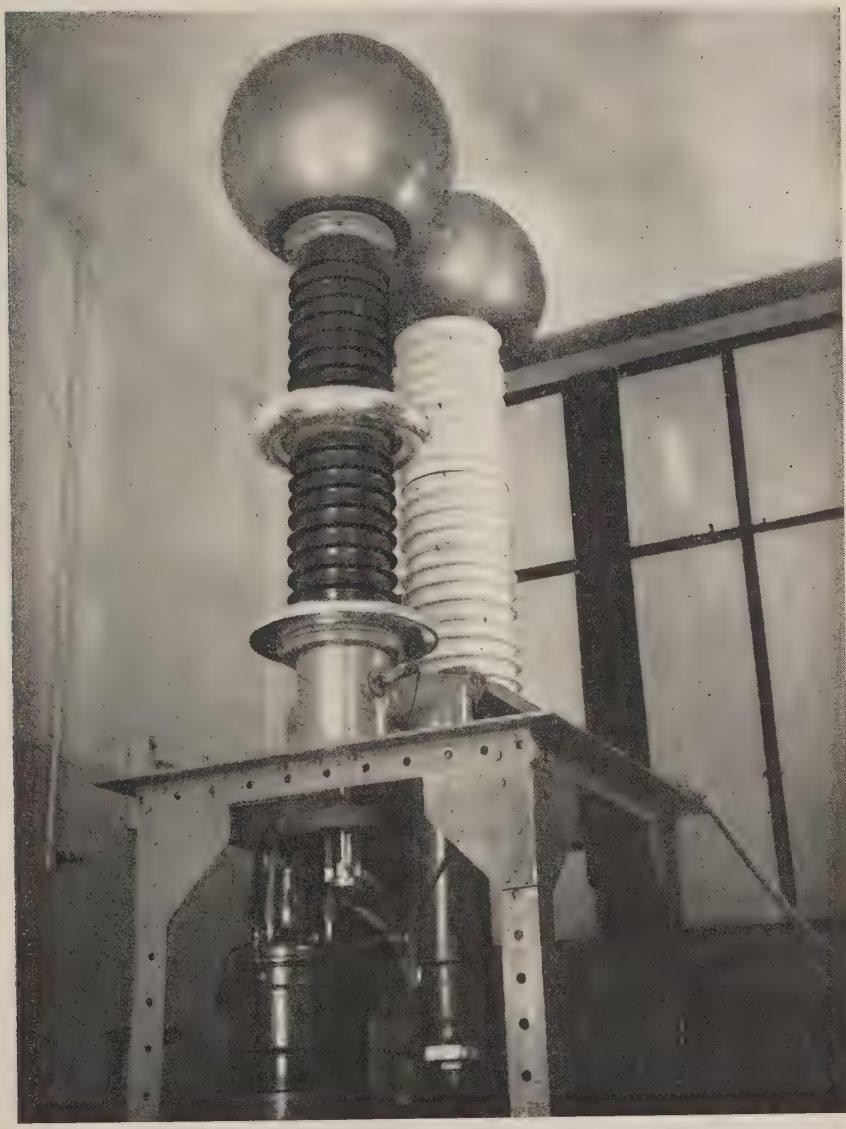

Abb. 14. – Van de Graaff Generator und dazugehörendes Beschleunigungsrohr für Elektronen. Aufgestellt für Studierzwecke im Laboratorium des Lehrstuhles für Theoretische Elektrizitätslehre der Technischen Universität in Budapest.

von 700 kV (siehe Abb. 14), welchen der ebenfalls unter der Führung des Verfassers stehende Lehrstuhl der Theoretischen Elektrizitätslehre der Technischen Universität ausschließlich zu Unterrichtszwecken in Betrieb gesetzt hat.

* * *

Ich egreife noch die Gelegenheit, um meinen Dank denjenigen auszusprechen, die durch ihre Mitwirkung das Zustandekommen der hier beschriebenen Anlagen gefördert haben. So waren J. ERÖ und G. SCHMIDT bei den ersten Beschleunigungsexperimenten tätig. Die Ionenquelle ist den Experimenten von J. ERÖ zu verdanken. I. MÉREY hat bei den Konstruktions- und Ausführungsarbeiten des Kaskadengenerators und des Tankgenerators, E. PÁSZTOR, J. ROÓSZ und Frau SIEGLER bei dem Zustandebringen der Beschleunigungsanlage, P. KOSTKA bei dem Bau und den Messungen des Tankgenerators mitgewirkt; G. KÁLMÁN und L. VARGA haben den 600 kV Generator mit Hochfrequenzheizung konstruiert.

Von den Außenmitarbeitern war G. VARGA derjenige, der vor dem Krieg den ersten Kaskadengenerator in Ungarn konstruierte: der Umformer und der Transformator sind seinem Apparat entnommen. Oberingenieur LENGYEL hat den Tank, Oberingenieur MARTON die Reinigungsanlage des 4 MV Apparates gebaut. Dank dem Erfahrungsaustausch mit dem Institut von Prof. SZALAY konnten wir an der Bandkonstruktion die nötigen Vervollkommnungen durchführen.

Ein besonderer Dank soll noch der Mechanischen und Schwachstromwerkstatt unserer Abteilung für die Verfertigung der Apparate ausgesprochen werden.

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SUPPLEMENTO AL VOLUME III, SERIE X
DEL NUOVO CIMENTO

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1° Semestre

Wechselwirkungskräfte und Renormalisation in der Photonpaar-Theorie.

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(ricevuto il 1° Febbraio 1956)

Inhalt: **1.** Einleitung. – **2.** Hamiltonfunktion. Masserenormalisation. – **3.** Eigenwerte. – **4.** Selbstenergie und Kräfte. Renormalisation. – **5.** Theorie der Streuung. – **6.** Renormalisationsfragen. Schluß. – Anhang I. Auswertung von $C(z)$ und $\Delta(z)$. – Anhang II. Heisenberg-Darstellung. – Anhang III. Eigenschaften der Matrizen $R_{()}$. – Anhang IV. S-Matrix für klassisches Strahlungsfeld.

1. – Einleitung.

Mit der Bezeichnung « Photonpaar-Theorie » soll die formale Zugehörigkeit der vorliegenden Arbeit zu den Feldtheorien mit festen Quellen endlicher Ausdehnung, bei denen die Wechselwirkungsenergie bilinear in den Feldvariablen ist, hervorgehoben werden. Insbesondere besteht eine enge Verwandtschaft zu der von Wentzel ([1,2]) untersuchten skalaren Mesonpaar-Theorie. Der Vektorcharakter des Feldes macht hier jedoch eine gewisse Verallgemeinerung gegenüber Wentzel notwendig, die wir gerade so vornehmen, daß die Anzahl und Anordnung der Quellen beliebig gelassen werden kann. Außerdem werden wir eine allgemeinere Hauptachsentransformation entwickeln, die, wie uns scheint, erst eine befriedigende Formulierung des Streuproblems in Schrödinger-Darstellung ermöglicht.

(1) G. Wentzel: *Zeits f. Phys.*, **118**, 277 (1941).
(2) G. Wentzel: *Helv. Phys. Acta*, **15**, 111 (1942).

Physikalisch ist die hier dargestellte Theorie eine stark vereinfachte Quantenelektrodynamik, so wie etwa die Arbeiten von PAULI und FIERZ [3] und von VAN KAMPEN [4]. Es wird die Wechselwirkung von schweren geladenen Partikeln ohne Spin mit Photonen in unrelativistischer Näherung behandelt (das Modell ist zuerst von KRAMERS untersucht worden, vergl. [4]). Während jedoch in den beiden zitierten Arbeiten ein äußeres Kraftfeld eine wesentliche Rolle spielt, haben wir, entsprechend der veränderten Zielsetzung, auf die Mitnahme äußerer Felder verzichtet. Dies hat den Vorteil, daß die Bloch-Nordsieck-Transformation, d.h. die Nullpunktverschiebung der Feldvariablen zur Elimination des linearen Wechselwirkungs-Terms, trivial wird. Trotz dieser Verschiedenheit des Standpunkts sind einige Züge der vorliegenden Theorie von VAN KAMPEN bereits vorweggenommen worden. In formaler Hinsicht ist allerdings die hier verwendete Impulsraum-Darstellung den von VAN KAMPEN benutzten und später von STEINWEDEL [5] übernommenen Dipol-Eigenfunktionen überlegen, weil letztere nämlich für mehr als zwei Quellen gar nicht mehr definiert werden können, während man im Falle von zwei Zentren auf elliptische Koordinaten übergehen muß und auf nicht geläufige Eigenwert-Gleichungen stößt. Andrerseits sind wir natürlich im Unterschied zu VAN KAMPEN nicht in der Lage, eine Formel für die Lamb-shift anzugeben.

Dem Wentzel'schen Modell wurde in jüngster Zeit wieder vermehrtes Interesse entgegengebracht, da es wegen der strengen Lösbarkeit als Prüfstein für grundsätzliche Fragen dienen kann. So haben MORPURGO and TOUSCHEK [6] die Brauchbarkeit der Tamm-Dancoff Methode untersucht. Ferner gaben KLEIN und MCCORMICK [7] eine Neuformulierung der Theorie im Sinne des Yang-Feldman Formalismus und streiften dabei auch die Frage der Renormalisation [8]. Das letztere Problem ist insbesondere Gegenstand von zwei gleichzeitigen Studien von THIRRING [9] und ARNOUS [10]. Die Anregung dazu war eine Arbeit von KÄLLÉN und PAULI [11], in welcher das Lee'sche Modell [12] einer kritischen Prüfung unterzogen worden war, mit dem Ergebnis, daß für Punktquellen die Renormalisation nicht durchführbar ist.

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[3] W. PAULI und M. FIERZ: *Nuovo Cimento*, **15**, 167 (1938).
[4] N. G. VAN KAMPEN: *Dan. Mat. Fys. Medd.*, **26**, no. 15 (1951).
[5] H. STEINWEDEL: *Ann. der Phys.*, **15**, 207 (1955).
[6] G. MORPURGO und B. F. TOUSCHEK: *Nuovo Cimento*, **10**, 1681 (1953).
[7] A. KLEIN und B. H. MCCORMICK: *Phys. Rev.*, **98**, 1428 (1955).
[8] Die Renormalisation der Mesonpaar-Theorie mit Punktquellen war schon früher Gegenstand einer Arbeit von L. VAN HOVE: *Physica*, **18**, 145 (1952).
[9] W. THIRRING: *Helv. Phys. Acta*, **28**, 344 (1955).
[10] E. ARNOUS: *Journ. Phys. et Rad.*, **17**, 107 (1956).
[11] G. KÄLLÉN und W. PAULI: *Dan. Mat. Fys. Medd.*, **30**, no. 7 (1955).
[12] T. D. LEE: *Phys. Rev.*, **95**, 1329 (1954).

Dieses bemerkenswerte Resultat gab auch der vorliegenden Arbeit einen wesentlichen Impuls. Dabei ist hervorzuheben, daß eine Photonpaar-Theorie gegenüber dem Wentzel'schen Fall vermöge ihrer Beziehung zur klassischen Elektrodynamik und wegen der verschwindenden Ruhemasse der Feldpartikel im Vorteil ist ([10,13]). Es besteht nämlich hier die Möglichkeit, mehrere unabhängige Vorschriften für die Renormalisation der Kopplungskonstanten bzw. der Ladung anzugeben, welche alle zum selben Resultat führen. Andrerseits kann man vielleicht hoffen, aus den Konsequenzen der Photonpaar-Theorie einige Hinweise für die allgemeine Quantenelektrodynamik zu gewinnen.

Im Lichte der letztgenannten Beziehung verdient besonders die Berechnung der « Photonen-Kräfte » zwischen geladenen Partikeln eine gewisse Beachtung. Denn die Reichweite einer solchen « Korrektur zum Coulomb'schen Gesetz » liefert einen Anhaltspunkt für die räumlichen Abmessungen, unterhalb derer mit Abweichungen von der Quantenelektrodynamik zu rechnen ist.

2. – Hamiltonfunktion. Masserenormalisation.

Wir betrachten das System, bestehend aus einer Anzahl Z geladener unrelativistischer Partikeln endlicher Ausdehnung, den « Elektronen », in Wechselwirkung mit ihrem elektromagnetischen Feld (keine äußeren Felder). Dieses wird beschrieben durch die Hamiltonfunktion ([14])

$$H = \frac{1}{2}\int\{\boldsymbol{E}^2 + (\mathrm{rot}\,\boldsymbol{A})^2\}\mathrm{d}^3x + \sum_{n=1}^{Z}\frac{1}{2m_0}(\boldsymbol{p}_n - e\boldsymbol{A}(n))^2\,, \tag{2.1}$$

wo m_0 die mathematische Masse und

$$\boldsymbol{A}(n) = \int \boldsymbol{A}(\boldsymbol{x})\cdot v(\boldsymbol{x} - \boldsymbol{q}_n)\,\mathrm{d}^3x \tag{2.2}$$

ist und der Formfaktor für die Ladungsverteilung im Elektron, $v(\boldsymbol{x})$, der Normierung

$$\int v(\boldsymbol{x})\,\mathrm{d}^3x = 1 \tag{2.3}$$

([13]) S. Deser, W. E. Thirring und M. L. Goldberger: *Phys. Rev.*, **94**, 711 (1954).

([14]) Wir benutzen durchwegs die Schrödinger-Darstellung, sowie Heaviside-Einheiten und $\hbar = c = 1$.

genügt. Aus (2.1) und den kanonischen Vertauschungsrelationen (V.R.)

$$i\,[-\boldsymbol{E}_j(\boldsymbol{x}), \boldsymbol{A}_{j'}(\boldsymbol{x}')] = \delta_{jj'}\delta^3(\boldsymbol{x}-\boldsymbol{x}') \tag{2.4}$$

$$i\,[p_{jn}, q_{j'n'}] = \delta_{jj'}\cdot\delta_{nn'} \tag{2.5}$$

folgen dann die Maxwell'schen Gleichungen in der Eichung mit verschwindendem skalarem Potential

$$-\dot{\boldsymbol{E}} \equiv i[H, -\boldsymbol{E}] = -\operatorname{rot}\operatorname{rot}\boldsymbol{A} + \boldsymbol{i}\,, \tag{2.6}$$

$$\dot{\boldsymbol{A}} \equiv i[H, \boldsymbol{A}] \quad = -\boldsymbol{E}\,, \tag{2.7}$$

wenn man den Strom in der symmetrisierten Form schreibt

$$\boldsymbol{i} = \sum_n \frac{e}{2m_0}\cdot\{\boldsymbol{p}_n - e\,\boldsymbol{A}(n), v(\boldsymbol{x}-\boldsymbol{q}_n)\}\,. \tag{2.8}$$

Zu diesen Gleichungen kommt als Nebenbedingung ([15])

$$(-\operatorname{div}\boldsymbol{E} + \varrho)\,\Psi(t) = 0\,, \tag{2.9}$$

wo

$$\varrho = \sum_n e v(\boldsymbol{x}-\boldsymbol{q}_n) \tag{2.10}$$

die Ladungsdichte ist. Man verifiziert leicht die Kontinuitätsgleichung

$$\operatorname{div}\boldsymbol{i} + \dot{\varrho} = 0\,, \tag{2.11}$$

aus welcher mit (2.6) die Veträglichkeit der Nebenbedingung folgt.

Der Übergang zur Impulsraum-Darstellung erfolgt gemäß den Gleichungen

$$\left\{\begin{aligned} \boldsymbol{A}(\boldsymbol{x}) &= V^{-\frac{1}{2}}\sum_{\lambda=0,\pm1}\sum_{\boldsymbol{k}} \boldsymbol{e}_{\lambda\boldsymbol{k}} Q_{\lambda\boldsymbol{k}} \exp[i\boldsymbol{k}\boldsymbol{x}] \\ -\boldsymbol{E}(\boldsymbol{x}) &= V^{-\frac{1}{2}}\sum_{\lambda=0,\pm1}\sum_{\boldsymbol{k}} \boldsymbol{e}_{\lambda\boldsymbol{k}} P_{\lambda\boldsymbol{k}} \exp[-i\boldsymbol{k}\boldsymbol{x}] \\ v(\boldsymbol{x}) &= V^{-1}\sum_{\boldsymbol{k}} v_{\boldsymbol{k}} \exp[-i\boldsymbol{k}\boldsymbol{x}]\,, \end{aligned}\right. \tag{2.12}$$

in denen V das Periodizitätsvolumen ist. Aus Gründen der Zweckmäßigkeit lassen wir den Polarisationsindex λ die Werte -1, 0, $+1$ durchlaufen und

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([15]) Vergl. G. Wentzel: *Quantentheorie der Wellenfelder* (Wien, 1943), § 17.

setzen

$$e_{0k} = \frac{k}{|k|}\,; \quad e_{+1,k} = e_{-1,-k}; \quad (e_{\lambda k}, e_{\lambda' k}) = \delta_{\lambda\lambda'}\,. \tag{2.13}$$

Für die Komponenten dieser Vektoren folgt dann die Beziehung

$$\sum_{\lambda=0,\pm1} e_{j\lambda k} e_{j'\lambda k} = \delta_{jj'}\,. \tag{2.13'}$$

Die Hermitezität der Feldoperatoren $-\boldsymbol{E}$, $\boldsymbol{A}$ verlangt

$$Q_{-\lambda,-k} = Q^+_{\lambda,k}\,; \qquad P_{-\lambda,-k} = P^+_{\lambda,k}\,; \qquad \lambda = \pm 1 \tag{2.14}$$

$$Q_{0,-k} = -Q^+_{0,k}\,; \qquad P_{0,-k} = -P^+_{0,k} \tag{2.15}$$

und die V.R. (2.4) werden

$$i[P_{\lambda k}, Q_{\lambda' k'}] = \delta_{\lambda\lambda'}\delta_{kk'}\,. \tag{2.16}$$

Ferner folgt aus der Realität von $v(x)$ bzw. aus (2.3)

$$v_{-k} = v_k^*\,; \qquad v_0 = 1\,. \tag{2.17}$$

Die Hamiltonfunktion geht über in

$$\begin{cases} H &= H_0 + H_{\text{long}} + H_{\text{int}} \\ H_0 &\equiv \frac{1}{2}\sum_{\lambda=\pm1}\sum_k (P_{\lambda k}P^+_{\lambda k} + |k|^2 Q^+_{\lambda k}Q_{\lambda k}) \\ H_{\text{long}} &\equiv \frac{1}{2}\sum_k P_{0,k}P^+_{0,k} \\ H_{\text{int}} &\equiv \sum_n \frac{1}{2m_0}\cdot\left(p_k - \frac{e}{\sqrt{V}}\cdot\sum_{\lambda=0,\pm1}\sum_k e_{\lambda k}Q_{\lambda k}v_k \exp[ikq_n]\right)^2 \end{cases} \tag{2.18}$$

und die Nebenbedingung (2.9) nimmt die Form an

$$(-i|k|P_{0k} + \varrho_k)\Psi = 0\,, \tag{2.19}$$

wo

$$\varrho_k \equiv V^{-\frac{1}{2}}\sum_n e v_k \exp[ikq_n]\,, \tag{2.20}$$

die Fourierkomponente der Ladungsdichte ist. Wegen (2.19) ist der longi-

tudinale Teil H_{long} mit der Coulomb-Energie

$$H_{\text{coul}} = \frac{1}{2} \sum_{\boldsymbol{k}} \frac{\varrho_{\boldsymbol{k}} \varrho_{-\boldsymbol{k}}}{|\boldsymbol{k}|^2} \tag{2.21}$$

äquivalent gemäß

$$H_{\text{long}} \Psi = H_{\text{coul}} \Psi \,.$$

Zur Elimination der longitudinalen Variablen aus H_{int} macht man eine kanonische Transformation ([15])

$$\overline{H} = e^{-S} H e^{S} \,; \qquad S = \sum_{\boldsymbol{k}} \frac{\varrho_{\boldsymbol{k}}}{|\boldsymbol{k}|} Q_{0\boldsymbol{k}} \,.$$

Das Resultat ist

$$\overline{H} = H_0 + H_{\text{coul}} + \overline{H}_{\text{int}} \tag{2.22}$$

mit

$$\overline{H}_{\text{int}} = e^{-S} H_{\text{int}} e^{S} = \sum_{n} \frac{1}{2m_0} \left(\boldsymbol{p}_n - \frac{e}{\sqrt{V}} \sum_{\lambda = \pm 1} \sum_{\boldsymbol{k}} \boldsymbol{e}_{\lambda \boldsymbol{k}} Q_{\lambda \boldsymbol{k}} v_{\boldsymbol{k}} \exp\left[i \boldsymbol{k} q_n \right] \right)^2 . \tag{2.23}$$

Nun führen wir die Dipol-Näherung ([3,4]) ein, indem wir in $\boldsymbol{A}(n)$ die Operatoren $\boldsymbol{q}_n$ durch die mittleren Aufenthaltsorte $\boldsymbol{r}_n$ ersetzen, d.h. die Wirkung der Strahlungsrückstöße auf das Feld vernachlässigen. Dies ist dann gut, wenn die Compton-Wellenlänge

$$\lambda_c = \frac{1}{m} \qquad (m\text{: physikalische Masse}) \tag{2.24}$$

klein ist gegen die Abschneide-Wellenlänge, welche hier durch die endliche Ausdehnung $v(\boldsymbol{x})$ der Elektronladung hereinkommt. Als Radius der Ladungsverteilung definiren wir

$$\frac{1}{a} = \frac{4\pi}{V} \sum_{\boldsymbol{k}} \frac{|v_{\boldsymbol{k}}|^2}{|\boldsymbol{k}|^2} , \tag{2.25}$$

sodaß die obige Bedingung lautet

$$\frac{\lambda_c}{a} \ll 1 \,. \tag{2.26}$$

Zwecks Festlegung der Schreibweise und zum Vergleich mit späteren Formeln geben wir noch explizite Ausdrücke für die Coulomb-Energie (2.21).

In obiger Näherung wird mit (2.20).

$$H_{\text{coul}} = \frac{1}{2} \cdot \frac{e^2}{V} \sum_{n} \sum_{n'} \sum_{\boldsymbol{k}} \frac{|v_{\boldsymbol{k}}|^2}{|\boldsymbol{k}|^2} \exp[i\boldsymbol{k}\boldsymbol{r}_{nn'}], \tag{2.27}$$

wo

$$\boldsymbol{r}_{nn'} \equiv \boldsymbol{r}_n - \boldsymbol{r}_{n'} \tag{2.28}$$

ist. Unter der Annahme einer kugelsymmetrischen Abschneidefunktion v_k, für die wir schreiben

$$|v_{\boldsymbol{k}}|^2 \equiv u(|\boldsymbol{k}|)\,, \tag{2.29}$$

sodaß

$$u(0) = 1\,; \qquad u(-t) = u(t)\,, \tag{2.29'}$$

und nach Übergang zum kontinuierlichen $\boldsymbol{k}$-Raum gemäß

$$\sum_{\boldsymbol{k}} \to \frac{V}{(2\pi)^3} \int \mathrm{d}^3 k, \tag{2.30}$$

erhält man für (2.27)

$$H_{\text{coul}} = Z \cdot E_{\text{coul}} + V_{\text{coul}}\,. \tag{2.31}$$

Hier ist

$$E_{\text{coul}} = \frac{e^2}{4\pi} \cdot \frac{1}{2a} \tag{2.32}$$

die Coulomb-Selbstenergie und

$$V_{\text{coul}} \equiv \frac{e^2}{4\pi} \cdot \sum_{\substack{n \\ n<n'}} \sum_{n'} \frac{2}{\pi} \int_0^\infty \mathrm{d}t \cdot u(t) \, \frac{\sin t r_{nn'}}{t r_{nn'}} \tag{2.33}$$

das Coulomb-Potential. In (2.32) wurde für a der mit (2.24) äquivalente Ausdruck

$$\frac{1}{a} = \frac{2}{\pi} \int_0^\infty u(t)\,\mathrm{d}t \tag{2.25'}$$

benutzt. Wir betrachten insbesondere die beiden Fälle

$$\frac{r_{nn'}}{a} \gg 1\,; \qquad \text{alle } n \neq n' \tag{2.34}$$

und

$$\frac{r_{nn'}}{a} \ll 1\,; \qquad \text{alle } n \neq n'\,. \tag{2.35}$$

Im ersten Fall kann man in (2.33) $u(t)$ durch 1 ersetzen und erhält

$$V_{\text{coul}} = \frac{e^2}{4\pi} \sum_{\substack{n \\ n<n'}} \sum_{n'} \frac{1}{r_{nn'}}\,. \tag{2.36}$$

Im zweiten Fall führt eine Entwicklung von $\sin tr_{nn'}/tr_{nn'}$ in 2. Näherung auf

$$V_{\text{coul}} = \left\{ Z(Z-1) - \frac{\beta}{3} \sum_{\substack{n \\ n<n'}} \sum_{n'} \frac{r_{nn'}^2}{a^2} \right\} \cdot E_{\text{coul}}\,, \tag{2.37}$$

mit

$$\beta \equiv a^3 \frac{2}{\pi} \int_0^\infty \mathrm{d}t \cdot u(t) \cdot t^2\,. \tag{2.38}$$

Vor der weiteren Umformung der Hamiltonfunktion (2.22, 23) legen wir eine vereinfachte Schreibweise fest, indem wir die zusammengezogenen Indices

$$\begin{cases} k \equiv \lambda, \boldsymbol{k}\,; & (\lambda = \pm 1) \\ \nu \equiv j, n\,; & (j = 1, 2, 3;\ n = 1, 2, \ldots, Z) \end{cases} \tag{2.39}$$

und die Größen

$$\omega_k \equiv |\boldsymbol{k}|\,, \tag{2.40}$$

$$g_0 \equiv \frac{e^2}{m_0 V}\,, \tag{2.41}$$

$$\alpha_{\nu,k} \equiv e_{j\lambda \boldsymbol{k}} v_{\boldsymbol{k}} \exp\left[i\boldsymbol{k}\boldsymbol{r}_n\right] = \alpha^*_{\nu,-\boldsymbol{k}} \tag{2.42}$$

$$A_{k,k'} \equiv \sum_{\nu} \alpha^*_{\nu k} \alpha_{\nu k'} \equiv \alpha^*_k \cdot \alpha_{k'} = A^*_{k',k} = A_{-k',-k} \tag{2.43}$$

einführen. Ferner definieren wir die folgende Operation der « Transposition » von Vektoren und Matrizen mit k-Komponenten

$$\begin{cases} (X^T)_k \equiv X_{-k} \\ (Y^T)_{kk'} \equiv Y_{-k',-k} \quad [16]. \end{cases} \tag{2.44}$$

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[16] Die Operation $^{+T}$ ergibt die Zeitumkehr, vergl. z. B. F. G. Chew und M. L. Goldberger: *Phys. Rev.*, **87**, 778 (1952).

Für ein Matrixprodukt gilt dann offensichtlich

$$(Y_1 Y_2)^T = Y_2^T Y_1^T . \tag{2.44'}$$

Damit lautet die Realitätsbedingung (2.14) in Vektorform

$$Q^T = Q^+ ; \qquad P^T = P^+ \tag{2.45}$$

und (2.42) bzw. (2.43) wird

$$\alpha_\nu^T = \alpha_\nu^* , \tag{2.42'}$$

bzw.

$$A = A^+ = A^T \tag{2.43'}$$

(es ist $(A^+)_{kk'} = A^*_{k'k}$). Die letzte Beziehung gilt offensichtlich auch für die Diagonalmatrix mit den Elementen (2.40)

$$\omega = \omega^+ = \omega^T \tag{2.40'}$$

und somit auch für die Matrix

$$B \equiv \omega^2 + g_0 A = B^+ = B^T . \tag{2.46}$$

Die V.R. (2.16) werden

$$i[P_k Q_{k'}] = \delta_{kk'} \tag{2.47}$$

und die Hamiltonfunktion (2.22, 23) lautet in Matrixschreibweise

$$\left\{ \begin{aligned} &H = H_0(P,Q) + H_1(P,Q) + H' , \\ &H_0(P,Q) = \tfrac{1}{2}(PP^+ + Q^- \omega^2 Q) , \\ &H_1(P,Q) \equiv \tfrac{1}{2} g_0 Q^+ A Q , \\ &H' \equiv -\sqrt{\frac{g_0}{m_0}} \sum_\nu p_\nu \alpha_\nu Q + \frac{1}{2m_0} \sum_\nu p_\nu^2 + H_{\text{coul}} . \end{aligned} \right. \tag{2.48}$$

Es ist zu beachten, daß die q_ν nicht mehr vorkommen und infolgedessen die p_ν Konstanten sind, die den Zustand charakterisieren. Aus diesem Grund kann man in trivialer Weise den Term $\sim Q$ in H' durch eine Nullpunktverschiebung der Feldvariablen wegschaffen [17]

$$Q = Q' + c ; \qquad P = P' . \tag{2.49}$$

---

[17] Wegen des Vorhandenseins eines äußeren Potentials ist diese Bloch-Nordsieck Transformation bei PAULI und FIERZ [3] und bei VAN KAMPEN [4] nicht trivial.

Mit

$$c^T = c^* \tag{2.50}$$

ist die Invarianz von (2.45) gewährleistet. Das Verschwinden des Terms $\sim Q$ in H' verlangt

$$c^* B - \sum_\nu p_\nu \alpha_\nu = 0 \,.$$

Nun folgt aus Abschnitt 3, daß B nicht singulär ist, sodaß die obige Gleichung aufgelöst wird durch

$$c = \sqrt{\frac{g_0}{m_0}}\, B^{-1} \sum_\nu p_\nu^+ a_\nu^* \,. \tag{2.51}$$

Einsetzen von (2.49, 51) in die Hamiltonfunktion (2.48) gibt

$$\begin{cases} \overline{H} = H_{\mathrm{Ph}} + H_{\mathrm{El}} \,, \\ H_{\mathrm{Ph}}(P', Q') \equiv H_0(P', Q') + H_1(P', Q') = \tfrac{1}{2}(P'P'^+ + Q'^+ B Q') \,, \\ H_{\mathrm{El}} \equiv \dfrac{1}{2m_0} \sum_\nu \sum_{\nu'} \{\delta_{\nu\nu'} - g_0 \alpha_\nu B^{-1} \alpha_{\nu'}^*\} p_\nu p_{\nu'} + H_{\mathrm{coul}} \,. \end{cases} \tag{2.52}$$

Es handelt sich nun zunächst darum, B^{-1} zu bestimmen. Dazu führen wir die Matrix ein

$$C(z) \equiv g_0 \sum_k \alpha_k \times \alpha_k^* \cdot \frac{1}{\omega_k^2 - z} \,. \tag{2.53}$$

Hier wurde im Unterschied zum skalaren Produkt der Vektoren α_k in (2.43) die $\times$-Multiplikation eingeführt, deren Bedeutung aus der Komponentenschreibweise

$$C_{\nu\nu'}(z) = g_0 \sum_k \frac{\alpha_{\nu k} \alpha_{\nu' k}^*}{\omega_k^2 - z} \tag{2.53'}$$

hervorgeht. Die Reziproke von (2.46) lautet jetzt

$$B_{kk'}^{-1} = \frac{1}{\omega_k^2} \delta_{kk'} - g_0 \frac{\alpha_k^*}{\omega_k^2} \cdot (1 + C(0))^{-1} \cdot \frac{\alpha_{k'}}{\omega_{k'}^2} \,, \tag{2.54}$$

wie leicht zu verifizieren ist. Weiter erhält man mit (2.53)

$$\begin{aligned} g_0 \alpha_\nu B^{-1} \alpha_{\nu'}^* &= \{C(0) - C(0)(1 + C(0))^{-1} C(0)\}_{\nu\nu'} \\ &= \{C(0)(1 + C(0)^{-1}\}_{\nu\nu'} \end{aligned}$$

und somit für den Elektronenanteil der Hamiltonfunktion (2.52)

$$H_{\mathrm{El}} = \frac{1}{2m_0} \sum_{\nu} \sum_{\nu'} (1 + C(0))^{-1}_{\nu\nu'} = p_\nu p_{\nu'} + H_{\mathrm{coul}} \,. \tag{2.55}$$

Aus der Forderung, daß dieser Ausdruck für unendliche Separation aller Elektronen die Form annehmen soll

$$H_{\mathrm{El}} = \frac{1}{2m} \sum_{n} \boldsymbol{p}_n^2 + H_{\mathrm{coul}} \qquad (r_{nn'} \to \infty) \tag{2.56}$$

wo m die physikalische Masse ist, erhält man eine Vorschrift für die Masserenormalisation. Nun folgt aus Anhang I, (I.17, 2), für $\boldsymbol{r}_{nn'} \to \infty$ (alle $n \neq n'$)

$$1 + C(z) = D_0(z) \cdot 1 \tag{2.57}$$

mit

$$D_0(z) = 1 + \frac{2}{3} g_0 \sum_{\boldsymbol{k}} \frac{|v_{\boldsymbol{k}}|^2}{|\boldsymbol{k}|^2 z - z} \,, \tag{2.58}$$

sodaß die Matrix in (2.55) diagonal wird und die Renormierungsvorschrift lautet [3,4]

$$m_0 \cdot D_0(0) = m \,. \tag{2.59}$$

Mit (2.29, 30) und der Definition

$$r_{\mathrm{co}} \equiv g_0 \frac{V}{6\pi} = \frac{2}{3} \cdot \frac{e^2}{4\pi} \cdot \frac{1}{m_0} \,, \tag{2.60}$$

($(3/2) r_{\mathrm{co}}$ ist der mathematische « klassische Elektronradius ») kann man auch schreiben

$$D_0(z) = 1 + r_{\mathrm{co}} \cdot \frac{2}{\pi} \int_0^\infty \frac{t^2 u(t)\, \mathrm{d}t}{t^2 - z} \,, \tag{2.58'}$$

ferner wegen (2.25′)

$$\frac{m}{m_0} = D_0(0) = 1 + \frac{r_{\mathrm{co}}}{a} \,, \tag{2.59'}$$

oder mit (2.32)

$$m = m_0 + \frac{4}{3} E_{\mathrm{coul}} \,. \tag{2.59''}$$

Der Wert $4/3 \cdot E_{\text{coul}}$ für die elektromagnetische Masse ist derselbe wie in der klassischen Theorie des Elektrons.

3. – Eigenwerte.

Im Folgenden haben wir uns nur noch mit der Hamiltonfunktion des Photonen-Feldes, H_{Ph} in (2.52), zu befassen. Der erste Schritt, nämlich die Hauptachsentransformation, geschieht am einfachsten ([18]) in folgender Weise (die neuen Feldvariablen sollen wieder mit P_k, Q_k bezeichnet werden)

$$\left\{ \begin{array}{l} P'^{+} = MP^{+} \\ Q' \;\; = MQ \, . \end{array} \right. \tag{3.1}$$

Die Erhaltung der V.R. (2.47) und der Realitätsbedingung (2.45) verlangt

$$MM^{+} = 1 \tag{3.2}$$

$$M^{T} = M^{+} . \tag{3.3}$$

Durch die Transformation soll erreicht werden

$$H_{\text{Ph}}(P', Q') = H'_{\text{Ph}}(P, Q) \equiv \tfrac{1}{2} \cdot (PP^{+} + Q^{+}\Omega^{2}Q) \, , \tag{3.4}$$

mit Diagonalmatrix Ω. Dies hat die Bedingungen

$$M^{+}M = 1 \tag{3.2'}$$

$$M^{+}BM = \Omega^{2} \tag{3.5}$$

zur Folge. Ferner erhält man aus der letzten Gleichung wegen (2.44', 46) und (3.3)

$$\Omega^{2} = (\Omega^{2})^{+} = (\Omega^{2})^{T} . \tag{3.6}$$

Mit der « Vollständigkeitsrelation » (3.2) und mit (2.43, 46) lautet (3.5) explizit

$$(\Omega_k^2 - \omega_{k'}^2) M_{k'k} = g_0 \alpha_{k'}^{*} \cdot \sum_{k''} \alpha_{k''} M_{k''k} \, . \tag{3.7}$$

([18]) Eine allgemeinere Transformation wird in Abschnitt **5** studiert.

Daraus folgt mit (2.53)

$$(1 + C(\Omega_k^2)) \cdot \sum_{k'} \alpha_{k'} M_{k'k} = 0 .$$

Somit ist die Lösung von (3.7)

$$M_{k'k} = g_0 \sum_{\nu} \alpha^*_{\nu k'} X_{\nu k} \cdot \frac{1}{\Omega_k^2 - \omega_{k'}^2} , \tag{3.8}$$

wo $X_{\nu k}$ diejenige geeignet normierte Lösung des Systems von $3Z$ linearen Gleichungen

$$\sum_{\nu'} (1 + C(z))_{\nu\nu'} X_{\nu'} = 0 \tag{3.9}$$

ist, die zur Wurzel Ω_k^2 der charakteristischen Gleichung

$$\Delta(z) \equiv \mathrm{Det} \,|1 + C(z)| = 0 \tag{3.10}$$

gehört.

Für die Existenz der obigen Hauptachsentransformation ist wegen (3.6) notwendig, daß (3.10) eine Folge von reellen Wurzeln Ω_k^2 hat, mit welcher die Lösungen (3.8) ein vollständiges System im Sinne von (3.2) bilden. Die Frage, ob außer dieser Folge noch komplexe Nullstellen erlaubt sind, können wir, wenigstens für den Fall $g_0 < 0$, nicht beantworten, da die Auswertung der Vollständigkeitsrelation (3.2) wohl kaum in allgemeiner Form möglich ist. Dagegen wird in Anhang III durch Übergang zum kontinuierlichen $\boldsymbol{k}$-Raum gezeigt, daß die Frage für den Fall $g_0 > 0$ zu verneinen ist.

Alles, was man auf Grund der allgemeinen Formeln schließen kann, ist daß die Wurzeln von $\Delta(z)$ in konjugiert komplexen Paaren auftreten. Wegen (2.42) gilt nämlich

$$C^+(z) = C(z^*) \tag{3.11}$$

(es ist $(C^+)_{\nu\nu'} \equiv C^*_{\nu'\nu}$) und somit

$$\Delta^*(z) = \Delta(z^*) . \tag{3.12}$$

Zur vollständigen Bestimmung der $X_{\nu k}$ ist noch (3.2′) zu berücksichtigen. Im Fall $\Omega_k^2 = \Omega_{k'}^2$ folgt mit (2.53)

$$(M^+M)_{kk'} = g_0 \sum_{\nu} \sum_{\nu'} X^*_{\nu k} \left[\frac{\mathrm{d}}{\mathrm{d}z} C_{\nu\nu'}(z)\right]_{z = \Omega_k^2} \cdot X_{\nu' k'} .$$

Dies ergibt für $k = k'$ die Normierung, für $k \neq k'$ eine Orthogonalisierung der $X_{\nu k}$. Der Fall $\Omega_k^2 \neq \Omega_k^2$ gibt mit (3.11)

$$(M^+M)_{kk'} = g_0 \sum_{\nu} \sum_{\nu'} X_{\nu k}^* [C(\Omega_k^2) - C(\Omega_{k'}^2)] X_{\nu' k'} \cdot (\Omega_k^2 - \Omega_{k'}^2)^{-1} .$$

Hier ist die rechte Seite wegen (3.9) identisch null, im Einklang mit (3.2′).

Eine explizite Diskussion der reellen Wurzeln von (3.10) ergibt folgendes Bild: Zwischen je zwei aufeinanderfolgenden ω_k^2-Werten liegt eine feste Anzahl N_z von Wurzeln ([19]), die wir alle dem kleineren ω_k^2-Wert zuordnen und mit Ω_k^2 bezeichnen wollen, ohne einen Extra-Index zu ihrer Unterscheidung einzuführen ([20]). Für $g_0 < 0$ tritt zudem eine Anzahl N_z' negativer Wurzeln auf ([21]), die wir zusammenfassend mit Ω_0 bezeichnen wollen ([20]).

Dieser Sachverhalt sei anhand der einfachsten Fälle illustriert. Für $Z-1$ ist nach (I.21′)

$$\Delta(z) = (D_0(z))^3 . \tag{3.13}$$

Aus (I.2) ist leicht zu sehen, daß nur reelle Wurzeln existieren, von denen wegen ($z = x + iy$)

$$\frac{\mathrm{d}}{\mathrm{d}x} D_0(x) = \frac{2}{3} g_0 \sum_{\boldsymbol{k}} \frac{|v_{\boldsymbol{k}}|^2}{(|\boldsymbol{k}|^2 - x)^2} \gtrless 0 \quad \text{für} \quad g_0 \gtrless 0 \tag{3.14}$$

genau eine (dreifach zu zählende, $N_z = 3$) zwischen je zwei aufeinanderfolgenden $|\boldsymbol{k}|^2$-Werten liegt. Aus Fig. 1 ist auch der isolierte Eigenwert $\Omega_0^2 < 0$ für $g_0 < 0$ ersichtlich ($N_z' = 3$). Im Fall $Z = 2$ gilt (I.21′) noch exakt; wir schreiben dafür mit Hilfe von (I.2, 7)

$$\Delta(z) = \prod_j D_{1j}(z) \cdot D_{2j}(z) , \tag{3.15}$$

wo

$$D_{\frac{1}{2}j}(z) = 1 + g_0 \sum_{\boldsymbol{k}} \frac{|v_{\boldsymbol{k}}|^2}{|\boldsymbol{k}|^2 - z} \left(\frac{2}{3} \pm f_j(|\boldsymbol{k}| r) \right) \tag{3.15′}$$

([19]) Eine Komplikation tritt ein bei «scharfem Abschneiden», d.h. wenn $v_{\boldsymbol{k}} = 0$ ist für $|\boldsymbol{k}|$ größer als der Abschneidewert, weil dann oberhalb dieses Wertes für $g_0 > 0$ nur noch eine, für $g_0 < 0$ keine Wurzel mehr kommt. Diesen Fall wollen wir ausschließen, Vergl. die Anm. 5 bei KÄLLÉN und PAULI ([11]).

([20]) Man fasse die $\Omega_k^2 (\Omega_0^2)$ selber noch als Diagonalmatrizen in einem N_z (N_z')-dimensionalen Raum auf. ω_k^2 ist dann als Multiplum der 1-Matrix und P_k, Q_k sind als Vektoren in diesem Raum anzusehen.

([21]) Ein derartiger unphysikalischer Eigenwert wurde zuerst von VAN KAMPEN ([4]) diskutiert (Kramers'sches Modell) und tauchte kürzlich auch bei der Renormierung des Lee'schen Modells ([11]) und des Wentzel'schen Modells ([9,10]) auf.

und $r \equiv r_{21}$ ist. Wegen (I.5″) gilt analog zu (3.14)

$$\frac{\mathrm{d}}{\mathrm{d}x} D_{\frac{1}{2}j}(x) = g_0 \sum_{\boldsymbol{k}} \frac{|v_{\boldsymbol{k}}|^2}{(|\boldsymbol{k}|^2 - x)^2} \left(\frac{2}{3} \pm f_j(|\boldsymbol{k}| r) \right) \gtrless 0 \quad \text{für} \quad g_0 \gtrless 0 \,. \tag{3.16}$$

$\Delta(z)$ hat also auch hier nur reelle Wurzeln, nämlich $N_z = 6$ zwischen je zwei aufeinanderfolgenden $|\boldsymbol{k}|^2$-Werten, von denen aber zweimal zwei zusammenfallen. Der Verlauf von $D_{\frac{1}{2}j}(x)$ ist wieder wie in Fig. 1. Für $g_0 < 0$ hat man $N'_z = 6$ isolierte Wurzeln $\Omega_0^2 < 0$.

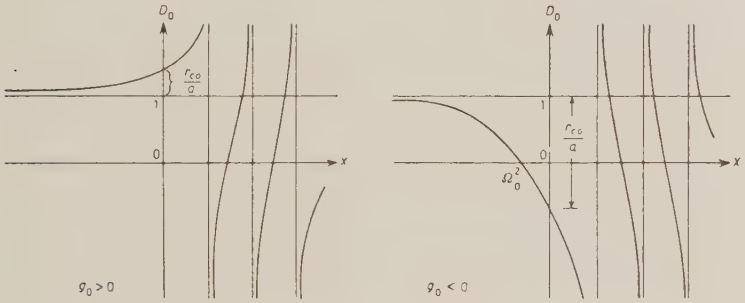

Fig. 1.

Fur $Z > 2$ wird die Diskussion sehr unübersichtlich. Wir beschränken uns daher auf die beiden Situationen (2.34) und (2.35). Im ersten Fall kann man (I.21′) analog zu (3.15) schreiben

$$\Delta(z) \cong (D_0(z))^{-3Z(Z-2)} \prod_{\substack{n \\ (n<n')}} \prod_{n'} \prod_{j} D_{1j}^{nn'}(z) D_{2j}^{nn'}(z) \tag{3.17}$$

mit

$$D_{\substack{1j \\ 2}}^{nn'}(z) \equiv 1 + g_0 \sum_{\boldsymbol{k}} \frac{|v_{\boldsymbol{k}}|^2}{|\boldsymbol{k}|^2 - z} \cdot \left(\frac{2}{3} \pm f_j(|\boldsymbol{k}| r_{nn'}) \right) \,. \tag{3.17′}$$

Dies gibt in jedem $|\boldsymbol{k}|^2$-Intervall $N_z = 3Z(Z-1)$ reelle Wurzeln Ω_k^2 (und nur solche) und ebensoviele isolierte Eigenwerte $\Omega_0^2 < 0$ für $g_0 < 0$. Für $r_{nn'} \to \infty$ (alle $n \neq n'$) wird die Näherung exakt. Gleichzeitig gehen die genäherten Wurzeln Ω^2 in die entsprechenden von $D_0(z)$ über. Man hat also in (I.21′) einen Grenzübergang, bei dem alle drei Ausdrücke $K_j^{nn'}(\Omega^2)$, $D_0(\Omega^2)$, $K_j^{nn'}(\Omega^2)/D_0(\Omega^2)$ gegen null konvergieren.

Im Falle sehr kleiner Abstände benutzen wir (I.23) in der Form

$$\Delta(z) \cong (1 + ZK_0(z))^2 \cdot \{1 + Z' K_0(z) + \delta a^2 z K_0^2(z)\}, \tag{3.18}$$

$$\delta = \sum_{\substack{n \\ (n<n')}} \sum_{n'} \frac{r_{nn'}^2}{a^2}; \qquad Z' = Z + \beta \frac{r_{co}}{a} \delta \,. \tag{3.18′}$$

Der erste Faktor ist von der Form $D_0(z)$, gibt also Wurzeln wie in Fig. 1. Die Nullstellen des zweiten Faktors können aufgefaßt werden als Schnittpunkte der Kurven $1 + Z' K_0(x)$ und $-\delta a^2 x K_0^2(x)$. Letztere ist für $x > 0$ stets negativ, für $x < 0$ positiv < 1 und geht für $x \to -\infty$ nach null. Da $K_0(x)$ monoton ist, folgt daß die Anzahl der Wurzeln des zweiten Faktors in jedem $|\boldsymbol{k}|^2$-Intervall 2, bzw. für die gebundenen Zustände ($g_0 < 0$) 1 beträgt. Somit ist hier $N_z = 4$, $N_z' = 3$.

Wir haben nun noch ein Wort zu sagen über die Konsequenzen der isolierten Wurzeln $\Omega_0^2 < 0$. Die Zuordnung von P-, Q-Variablen zu den Ω_0^2 hat natürlich so zu erfolgen, daß (2.45, 47) erfüllt ist. Doch ist die Anzahl der Variablenpaare weitgehend willkürlich. Am einfachsten ist es, jedem Ω_0^2 ein einziges Paar P_0, Q_0 mit den Eigenschaften

$$P_0^+ = P_0 \,; \qquad Q_0^+ = Q_0 \tag{3.19}$$

$$i\,[P_0 Q_0] = 1 \tag{3.20}$$

zuzuordnen. Der Beitrag zur Hamiltonfunktion (3.4) ist dann die indefinite Form

$$\tfrac{1}{2}(P_0^2 - |\Omega_0^2| Q_0^2)\,. \tag{3.21}$$

Dieser Operator besitzt nun, wie ARNOUS [10] gezeigt hat, entweder kein vollständiges System von Eigenfunktionen, sodaß die Quantisierung des «Oszillators» Q_0 schleierhaft ist, oder imaginäre Eigenwerte.

Eine andere Möglichkeit, die wir noch kurz diskutieren wollen, ist die Zuordnung von zwei Variablenpaaren $P_{\pm 0}$, $Q_{\pm 0}$. (2.45, 47) lautet jetzt

$$P_{-0}^+ = P_{+0} \,; \qquad Q_{-0}^+ = Q_{+0} \tag{3.22}$$

$$i\,[P_{\pm 0}, Q_{\pm 0}] = 1 \tag{3.23}$$

und der Beitrag zur Hamiltonfunktion (3.4) wird

$$P_{+0} P_{+0}^+ - |\Omega_0^2| Q_{+0}^+ Q_{+0}\,, \tag{3.24}$$

also wieder ein indefiniter Ausdruck. Durch die Transformation

$$\left\{\begin{aligned} P_{+0}^+ &= \frac{\sqrt{|\Omega_0|}}{2}\left\{\frac{1+i}{2}\,p_1 + \frac{1-i}{2}\,p_2 + (1+i)q_1 - (1-i)q_2\right\}, \\ Q_{+0} &= \frac{1}{2\sqrt{|\Omega_0|}}\left\{\frac{1+i}{2}\,p_1 + \frac{1-i}{2}\,p_2 + (1+i)q_1 - (1-i)q_2\right\} \end{aligned}\right. \tag{3.25}$$

erreicht man

$$p_r^+ = p_r\,; \qquad q_r^+ = q_r\,; \qquad (r = 1, 2) \tag{3.26}$$

$$i\,[p_r q_{r'}] = \delta_{rr'}\,. \tag{3.27}$$

Ferner geht (3.24) über in

$$|\,\Omega_0\,|\,(q_1 p_1 - q_2 p_2) = -\,i\,|\,\Omega_0\,|\cdot\left(q_1\,\frac{\partial}{\partial q_1} - q_2\,\frac{\partial}{\partial q_2}\right). \tag{3.24'}$$

Die Eigenwert-Gleichung zu diesem Operator ist natürlich separierbar. Durch Superposition erhält man als allgemeine Lösung zum Eigenwert λ

$$\psi_\lambda(q_1, q_2) = f(q_1\cdot q_2)\cdot\exp\left[\frac{i}{2}\,\frac{\lambda}{|\,\Omega_0\,|}\,\log\frac{q_1}{q_2}\right], \tag{3.28}$$

wie durch Einsetzen in die Gleichung leicht zu verifizieren ist. Dabei hat die Funktion f der Variable $q_1\cdot q_2$ lediglich differenzierbar zu sein, ist aber sonst willkürlich und λ kann irgend eine komplexe Zahl sein! Die Folgerungen aus dem Ansatz (3.24) sind also noch bedenklicher als diejenigen aus (3.21). Ein ähnlicher Fall wie (3.24) wurde übrigens von Pais und Uhlenbeck [22] früher schon diskutiert.

4. – Selbstenergie und Kräfte. Renormalisation.

Die Diagonalisierung der Hamiltonfunktion (3,4) des Photonen-Feldes geschieht in bekannter Weise durch Übergang auf Emissions- und Absorptionsoperatoren gemäß

$$\begin{cases} P_k = (\Omega_k/2)^{\frac{1}{2}}\,i\,(a_k^+ - a_{-k}) \\ Q_k = (2\Omega_k)^{-\frac{1}{2}}\,(a_k + a_{-k}) \end{cases} \tag{4.1}$$

mit den V.R.

$$[a_k a_{k'}^+] = \delta_{kk'}\,. \tag{4.1'}$$

Damit geht (3.4) über in

$$H'_{\mathrm{Ph}}(P, Q) = \sum_k (a_k^+ a_k + \tfrac{1}{2})\Omega_k\,. \tag{4.2}$$

Das Vacuum der freien ($g_0 = 0$) bzw. « physikalischen » ($g_0 \neq 0$) Photonen ist

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[22] A. Pais und G. E. Uhlenbeck: *Phys. Rev.*, **79**, 145 (1950).

zu definieren als der tiefste Zustand von $H_0(P', Q')$ bzw. $H'_{\mathrm{Ph}}(P, Q)$, also nach (4.2)

$$\left\{\begin{aligned} H_0(P', Q')\,|\,0'\rangle &= \tfrac{1}{2}\sum_k \omega_k\cdot|\,0'\rangle \\ H'_{\mathrm{Ph}}(P, Q)\,|\,0\rangle &= \langle\tfrac{1}{2}\sum_k \Omega_k\cdot|\,0\rangle\,. \end{aligned}\right. \tag{4.3}$$

Die physikalische Vacuum-Energie ist dann

$$\left\{\begin{aligned} \langle H_{\mathrm{Ph}}\rangle_0 &\equiv \langle 0\,|\,H'_{\mathrm{Ph}}(P, Q)\,|\,0\rangle - \langle 0'\,|\,H_0(P', Q')\,|\,0'\rangle \\ &= \tfrac{1}{2}\sum_k (\Omega_k - \omega_k)\,. \end{aligned}\right. \tag{4.4}$$

Es muß aber betont werden, daß die obige Definition des physikalischen Vacuums nur sinnvoll ist, wenn H'_{Ph} positiv definit ist. Für $g_0 < 0$ ist dies aber wegen der Existenz der Eigenwerte $\Omega_0^2 < 0$ gerade nicht der Fall. Trotzdem kann man natürlich den Ausdruck (4.1) bilden, und wir werden ihn im Folgenden sowohl für $g_0 > 0$ wie für $g_0 < 0$ diskutieren. Auf die physikalische Bedeutung im Fall $g_0 < 0$ kommen wir in Abschnitt 6 noch zu sprechen.

Für die Auswertung von (4.4) benutzen wir die Wentzel'sche Methode [23]. Man schreibt die Determinante (3.10) in der Form

$$\Delta(z) = \prod_k (z - \Omega_k^2) \,/\, \prod_k (z - \omega_k^2)\,, \tag{4.5}$$

wobei die Multiplizität der Nullstellen Ω_k^2 und Pole ω_k^2 [20] zu beachten ist. Dann gilt für jede Funktion $f(z)$, die innerhalb des Weges von Fig. 2 regulär und eindeutig ist,

$$\sum_k [f(\Omega_k^2) - f(\omega_k^2)] = \frac{1}{2\pi i}\int_\Gamma \mathrm{d}z\, f(z)\,\frac{\mathrm{d}}{\mathrm{d}z}\log \Delta(z)\,,$$

oder nach partieller Integration, und mit

$$\Delta_{(\pm)}(x) \equiv \lim_{\varepsilon\to 0} \Delta(x \pm i\varepsilon)\,, \tag{4.6}$$

$$\sum_k [f(\Omega_k^2) - f(\omega_k^2)] = \frac{1}{2\pi i}\lim_{\delta\to 0}\int_\delta^\infty \mathrm{d}x\, f'(x)\,\log\frac{\Delta_{(+)}(x)}{\Delta_{(-)}(x)}\,.$$

Setzt man hier

$$f(z) = \tfrac{1}{2}\cdot\sqrt{z}\,,$$

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[23] Vergl. G. Wentzel [2]. Eine andere Methode, die den diskreten k-Raum vermeidet und somit nur für $g_0 > 0$ anwendbar ist, findet sich bei Klein und McCormick [7].

so erhält man nach Substitution von x durch t^2

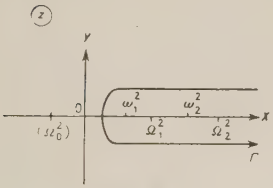

Fig. 2.

$$\langle H_{\mathrm{Ph}}\rangle_0 = \frac{1}{4\pi i}\int_0^\infty \mathrm{d}t\, \log \frac{\Delta_{(+)}(t^2)}{\Delta_{(-)}(t^2)}\,. \tag{4.7}$$

Dieser Formel stellen wir den entsprechenden Ausdruck in 2. störungstheoretischer Näherung gegenüber:

$$\begin{cases} \langle H_{\mathrm{Ph}}\rangle_0^{S.} = E^{(1)} + E^{(2)} \\ E^{(1)} \equiv \langle 0'|\, H_1(P', Q')\,|0'\rangle = \tfrac{1}{4} g_0\, \mathrm{Spur}\,(A\omega^{-1}) \\ E^{(2)} \equiv -\sum_k \sum_{k'} |\langle 1'_k 1'_{k'}|\, H_1(P', Q')\,|0'\rangle|^2/(\omega_k + \omega_{k'}) \\ \quad = -\tfrac{1}{8} g_0^2 \sum_k \sum_{k'} |A_{k,-k'}|^2/\omega_k \omega_{k'}(\omega_k + \omega_{k'})\,. \end{cases} \tag{4.7'}$$

Dabei ist $|1'_k 1'_{k'}\rangle$ der normierte Eigenzustand von $H_0(P', Q')$ mit zwei freien Photonen, (k), (k').

Für $Z = 1$ ist (4.7) die Selbstenergie E_0 eines Elektrons im Photonen-Feld, und man erhält mit (3.13)

$$E_0 = \frac{3}{4\pi i}\int_0^\infty \mathrm{d}t \log \frac{D_{0(+)}(t^2)}{D_{0(-)}(t^2)}\,. \tag{4.8}$$

Hier ist $D_{0(\pm)}(t^2)$ wie in (4.6) definiert und gibt mit (2.58′)

$$D_{0(\pm)}(t^2) \equiv 1 + K_{0(\pm)}(t^2) = D_{0(1)}(t^2) \pm i r_{c0}\cdot |t|\cdot u(|t|) \tag{4.9}$$

wo

$$D_{0(1)}(t^2) = \frac{1}{2}(D_{0(+)} + D_{0(-)}) = 1 + r_{c0}\frac{2}{\pi}\int_{0\,P}^{\infty} \mathrm{d}t'\, \frac{t'^2 u(t')}{t'^2 - t^2} \tag{4.9'}$$

ist und P den Hauptwert bedeutet. Mit (4.9) wird die Selbstenergie

$$E_0 = \frac{3}{2\pi}\int_0^\infty \mathrm{d}t\ \mathrm{arctg}\ \frac{r_{c0}\cdot t\cdot u(t)}{D_{0(1)}(t^2)}\,. \tag{4.8'}$$

Nun geht $D_{0(1)}(t^2)$ für $t \to \infty$ gegen 1, sodaß E_0 divergiert, falls $u(t)$ nicht stärker als t^{-2} abfällt. Geht der Abfall mindestens wie t^{-n} $(n > 3)$, so gilt die

Abschätzung

$$D_{0(1)}(t^2) = 1 - \frac{r_{c0}}{a}\left[\frac{\beta}{(at)^2} + 0\left(\frac{1}{(at)^{n-1}}\right)\right],$$

wo β, (2.38), von der Größenordnung 1 ist. Daraus folgt im Falle $r_{c0}/a)^{\frac{1}{2}} \gg 1$, daß $D_{0(1)}$ in der Gegend

$$t \cong \frac{1}{a}\sqrt{\beta\,\frac{r_{c0}}{a}}$$

eine einfache Nullstelle $t = t_0$ hat, sodaß der arctg in (4.8′) für $1/a \lesssim t < t_0$ im Zweig $(\pi/2 \ldots 3\pi/2)$ liegt (die Zweige sind so festzusetzen, daß arctg $(\,) \to 0$ für $t \to \infty$). Somit gibt t_0 die Größenordnung von (4.8′),

$$E_0 \cong \text{const}\,\frac{1}{a}\sqrt{\frac{r_{c0}}{a}}\,;\quad \frac{r_{c0}}{a} \gg 1 \quad (^{24})\,. \tag{4.10}$$

Für $g_0 < 0$ ist aber (4.10) nicht mehr wahr, da dann, ganz abgesehen von den Eigenwerten Ω_0^2, die Nullstelle t_0 nicht existiert.

Im andern Extremfall, $r_{c0}/a \ll 1$, erhält man aus (4.8′)

$$E_0 = -\frac{3}{2\pi}\cdot r_{c0}\int_0^\infty \mathrm{d}t\cdot t\cdot u(t) \quad \frac{3}{2\pi^2}\cdot r_{c0}^2\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}t'\,\frac{tt'\,u(t)\,u(t')}{t+t'}\,;\qquad \frac{r_{c0}}{a} \ll 1\,. \tag{4.10′}$$

Dieser Ausdruck ist identisch mit der 2. störungstheoretischen Näherung (4.7′) für $Z = 1$.

Zur Illustration des gesamten Verlaufs geben wir die Ausrechnung für den einfachsten Fall, bei dem (4.8′) konvergiert, nämlich dür die Abschneidefunktion (I.13). Mit (I.15) wird, da wegen (I.12)

$$\sqrt{t^2 \pm i\varepsilon} = \pm|t| + i\varepsilon' \tag{4.11}$$

zu nehmen ist,

$$D_{0(\pm)}(t^2) = 1 + \frac{r_{c0}}{a}\left(1 - \frac{a^2}{4}t^2 \pm ia\,|t|\right)\Big/\left(1 + \frac{a^2}{4}t^2\right)^2. \tag{4.12}$$

---

(24) Dieses Resultat findet sich schon bei H. A. Bethe: *Phys. Rev.*, **55**, 681 (1939). Ferner gilt (4.10) auch für die skalare Mesonpaar-Theorie, wie man aus Formel (8) bei Wentzel (2) leicht sieht. Demgegenüber fanden Wigner, Critchfield und Teller: *Phys. Rev.*, **56**, 530 (1939) für die Elektronpaar-Theorie eine lineare Abhängigkeit der Selbstenergie von der Kopplungskonstante, wenn diese genügend groß ist.

Daraus folgt

$$E_0 = \frac{3}{\pi a} \cdot F_0\left(\frac{a}{r_{c0}}\right), \tag{4.13}$$

mit

$$F_0(\xi) \equiv \int_0^\infty \mathrm{d}\tau \operatorname{arctg} \frac{2\tau}{\xi(1+\tau^2)^2 + (1-\tau^2)} \,. \tag{4.14}$$

Bei der Auswertung dieses Integrals hat man den Verzweigungspunkten die nötige Beachtung zu schenken. Das Resultat ist

$$F_0(\xi) = \frac{1}{\sqrt{\xi}} \operatorname{arctg} \frac{2\sqrt{\xi}}{\xi - 1} - \log \frac{1+\xi}{\xi} \,; \qquad \xi > 0 \tag{4.14'}$$

wobei der arctg stetig zwischen π ($\xi = 0$) und 0 ($\xi \to \infty$) zu variieren ist, und

$$F_0(\xi) = -\frac{1}{\sqrt{|\xi|}} \log \left| \frac{1+\sqrt{|\xi|}}{1-\sqrt{|\xi|}} \right| - \log \left| \frac{1-|\xi|}{\xi} \right| \,; \qquad \xi < 0 \,. \tag{4.14''}$$

Der Verlauf von $F_0(\xi)$ ist in Fig. 3 wiedergegeben (man beachte die logarithmische Singularität der Ableitung im Punkt $\xi = -1$).

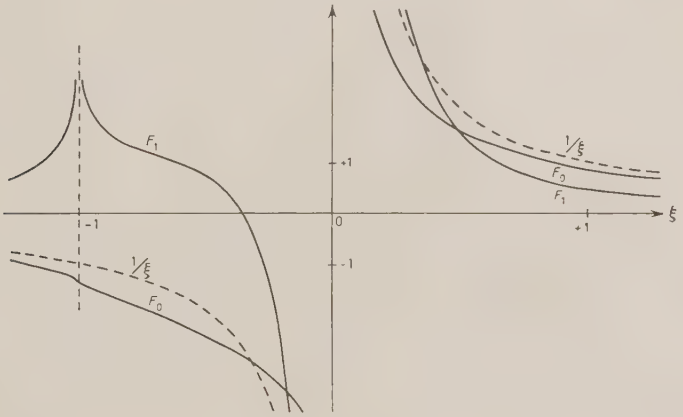

Fig. 3.

Sind Z (>1) Elektronen vorhanden, so üben diese infolge ihrer Kopplung an das Photonen-Feld gegenseitig Kräfte aus, die ebenfalls durch (4.4), als Potential, beschrieben werden. Durch Abziehen der Z Selbstenergien erreicht man, daß dieses Potential für unendliche Separation aller Elektronen ver-

schwindet. Mit (4.7, 8) wird

$$W(r_{nn'}) = \langle H_{\mathrm{Ph}}\rangle_0 - Z\cdot E_0 = \frac{1}{4\pi i}\int_0^\infty \mathrm{d}t \left[\log \frac{\Delta_{(+)}(t^2)}{(D_{0(+)}(t^2))^{3Z}} - \log \frac{\Delta_{(-)}(t^2)}{(D_{0(-)}(t^2))^{3Z}}\right]. \tag{4.15}$$

Als ersten Fall wollen wir die Zwei-Körper Kräfte für großen Abstand berechnen ([25]). Mit (I.21′), was für $Z = 2$ eine exakte Formel ist, erhält man nach Weglassen der Indices $n = 1$, $n' = 2$

$$W(r) = \frac{1}{4\pi i}\int_0^\infty \mathrm{d}t \sum_j \left\{\log\left[1 - \left(\frac{K_{j(+)}(t^2)}{D_{0(+)}(t^2)}\right)^2\right] - \log\left[1 - \left(\frac{K_{j(-)}(t^2)}{D_{0(-)}(t^2)}\right)^2\right]\right\}. \tag{4.16}$$

Dabei sind die K_j durch (I.11) gegeben und das Koordinatensystem ist so festgelegt, daß die 3. Achse die Richtung von $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ hat. Für $r/a \gg 1$ wird mit (I.16) und (4.11)

$$K_{j(\pm)}(t^2) = 3i\cdot\frac{r_{c0}}{r}\cdot\frac{\varphi_j(\pm|t|r)}{\pm|t|r}.$$

Aus der Definition (I.8) der Funktionen φ_j ist ersichtlich, daß $K_{j(+)}$ und $K_{j(-)}$ komplex konjugiert sind. Für $tr \gtrsim r/a \gg 1$ enthalten diese Ausdrücke einen sehr schnell oszillierenden Term und einen solchen, der wie $(tr)^{-2}$ abfällt. Daher trägt das Intervall $t \gtrsim 1/a$ wenig bei zum Integral (4.16) und man begeht einen kleinen Fehler, wenn man für alle t den für $t \ll 1/a$ gültigen Wert von $D_{0(\pm)}(t^2)$ einsetzt. Für diesen gilt

$$D_{0(\pm)}(t^2) = 1 + \frac{r_{c0}}{a}\left(1 + 0(at)\right).$$

Damit folgt aber, unter der Voraussetzung

$$\frac{1}{r} \ll \left|\frac{1}{r_{c0}} + \frac{1}{a}\right| \tag{4.17}$$

(welche für $r_{c0} < 0$ stärker als $r/a \gg 1$ ist), wegen (I.8″)

$$\left|\frac{K_{j(\pm)}}{D_{0(\pm)}}\right| \ll 1. \tag{4.18}$$

([25]) Eine ähnliche Rechnung für den Fall der Mesonpaar-Theorie findet sich außer bei Wentzel ([2]) in der Arbeit von W. Pauli und N. Hu: *Rev. Mod. Phys.*, **17**, 267 (1945), § 4.

(4.16) wird jetzt

$$W(r) = -\frac{1}{2\pi}\,\mathrm{Im}\int_0^\infty \mathrm{d}t\,\sum_j \left(\frac{K_{j(+)}(t^2)}{D_{0(+)}(t^2)}\right)^2,$$

oder eingesetzt

$$W(r) = -\frac{9}{4\pi}\cdot I\cdot\left(\frac{1}{r_{c0}}+\frac{1}{a}\right)^{-2}\cdot\frac{1}{r^3}, \tag{4.19}$$

wo

$$I = \mathrm{Im}\int_0^\infty \mathrm{d}\tau\left\{\left(e^{i\tau}+i\,\frac{e^{i\tau}}{\tau}-\frac{e^{i\tau}-1}{\tau^2}\right)^2+2\left(i\,\frac{e^{i\tau}}{\tau}-\frac{e^{i\tau}-1}{\tau^2}\right)^2\right\}. \tag{4.20}$$

Demgegenüber liefert die 2. störungstheoretische Näherung (4.7′) für $Z=2$ und $r/a\gg 1$ (der Gang der Rechnung ist wie im Anhang der ersten Wentzel'schen Arbeit [1]), doch verursacht der Vektorcharakter des Feldes beträchtliche Mehrarbeit, da man hier zusätzlich über die Polarisationsindices λ zu summieren hat und die Winkelabhängigkeit nicht mehr trivial ist)

$$W^{s.}(r) = -\frac{9}{4\pi}\cdot I\cdot\frac{r_{c0}^2}{r^3}. \tag{4.21}$$

Dies ist identisch mit dem ersten Term einer Entwicklung von (4.19) nach Potenzen von r_{c0}/a. Endlich gibt die Auswertung des Integrals (4.20)

$$I = \tfrac{1}{2}. \tag{4.20′}$$

Die Resultate (4.19, 21) versetzen uns in die Lage, nun auch die Renormalisation der Ladung e studieren zu können. Dazu stehen uns zwei unabhängige Vorschriften zur Verfügung, die beide auf ein identisches Resultat führen. Die naheliegendste Renormierungsvorschrift ergibt sich aus einer Messung der Coulomb-Kraft für großen Abstand r. Da (4.19) für $r\to\infty$ stärker abfällt als das Coulomb-Potential, folgt direkt, daß man gar keine Ladungsrenormalisation hat (wenigstens für $g_0>0$ nicht). Dies steht im Einklang mit der Tatsache, daß in der vorliegenden Theorie wegen ihres unrelativistischen Charakters (Energie positiv definit für $g_0>0$) keine Vacuumpolarisation existiert. Man kann also jetzt analog zu (2.59) den physikalischen «klassischen Elektronradius» definieren,

$$r_c \equiv \frac{2}{3}\cdot\frac{e^2}{4\pi}\cdot\frac{1}{m} = \frac{2}{3}\cdot\frac{1}{137}\cdot\lambda_c. \tag{4.22}$$

Hier wurde auf der rechten Seite

$$\frac{e^2}{4\pi} = \frac{1}{137}\,, \tag{4.23}$$

sowie die in (2.24) eingeführte Compton-Wellenlänge benutzt. Aus (2.59′) folgt dann die Beziehung

$$\frac{1}{r_c} = \frac{1}{r_{c0}} + \frac{1}{a}\,. \tag{4.24}$$

Als zweite Vorschrift kann man verlangen, daß (4.19) für $r_{c0} \to 0$ in die renormierte störungstheoretische Formel (4.21) (r_{c0} durch r_c ersetzt) übergeht. Daraus folgt wieder (4.24) und man kann mit Hilfe von (4.20′) für (4.19) schreiben

$$W(r) = -\frac{9}{8\pi}\cdot\frac{r_c^2}{r^3} \quad [26]\,. \tag{4.19′}$$

Zur Herleitung von (4.19) mußte (4.17) vorausgesetzt werden. Setzt man jetzt (4.24) in (4.17) ein, so folgt

$$\frac{r}{r_c} \gg 1\,. \tag{4.17′}$$

Dies ist die einzige Voraussetzung über die Stärke der Kopplung. Insbesondere bleibt (4.19′) sinnvoll, wenn man bei festem r_c zum Punktelektron ($a \to 0$) übergeht, während dieser Übergang bei festem r_{c0} gemäß (4.19) ein verschwindendes Potential geben würde (letzteres ist von Wentzel [1] betont worden). Auf diesen interessanten Zug der Theorie werden wir in Abschnitt 6 zurückkommen.

Die Bedeutung des Resultats (4.19′) wird besonders durchsichtig, wenn man es mit dem Coulomb-Potential (2.36) vergleicht. Mit (4.22) wird

$$\frac{W}{V_{\text{coul}}} = -\left(\frac{7.0\cdot r_c}{r}\right)^2 = -\left(\frac{0.034\,\lambda_c}{r}\right)^2\,. \tag{4.25}$$

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[26] Ähnliche Ausdrücke, aber für makroskopische, leitende oder polarisierbare Körper verschiedener Form wurden von H. B. G. Casimir und D. Polder: *Phys. Rev.*, **73**, 360 (1948); H. B. G. Casimir: *Journ. Chimie Phys.*, **46**, 407 (1949) und *Proc. Kon. Ned. Akad. v. Wet.*, B **51**, 793 (1948), sowie von H. G. van Bueren: *Proc. Kon. Ned. Akad. v. Wet.*, B **55**, 493 (1952) abgeleitet. Abgesehen von der ersten Arbeit wurde eine elegante Methode von Casimir benützt, die effektiv mit der hier gebrauchten Wentzel'schen übereinstimmt. Casimir berechnet die Differenz der Nullpunktsenergien eines Strahlungshohlraums mit und ohne Körper mittels klassicher Elektrodynamik.

Die Korrektur zum Coulomb'schen Gesetzt, W, beginnt sich somit für Abstände bemerkbar zu machen, die zwischen r_c und der Compton-Wellenlänge λ_c liegen. In diesem Bereich ist nun zwar die Voraussetzung (2.26) für die Vernachlässigbarkeit der Rückstöße unvereinbar mit $r/a \gg 1$, sodaß die Korrektur durch (4.25) sicher nicht quantitativ wiedergegeben wird. Doch steht die Größenordnung des Abstandes, bei dem die Korrektur merkbar wird, in Übereinstimmung mit einem (zwar noch unsichern) Resultat von SALECKER [27], der durch Analyse von Streuexperimenten mit Elektronen zum Schluß kommt, daß die Quantenelektrodynamik für Abstände kleiner als 6 bis $9r_c$ zu modifizieren ist.

Endlich ist noch darauf hinzuweisen, daß die Selbstenergie E_0 durch die Masserenormalisation (4.24) keineswegs eliminiert wird. Auch dies ist auf Grund des unrelativistischen Charakters der Theorie zu verstehen, da in diesem Fall zwischen Energie und Masse keine Identität besteht. Zudem bleibt ja auch die Coulomb-Selbstenergie (2.32) stehen. Ein Vergleich der beiden Selbstenergien zeigt übrigens, daß diese sich niemals kompensieren, denn (vergl. (4.13), (I.14))

$$\frac{E_0}{E_{\text{coul}}} = \frac{6}{\pi} \cdot 137 \cdot F_0\left(\frac{a}{r_c} - 1\right) \tag{4.26}$$

wird erst bei $a/r_c \cong -260$ gleich -1.

Als zweiten Fall wollen wir nun noch die Mehrkörper-Kräfte in der Näherung (2.35) studieren. Einsetzen von (I.23) in (4.15) gibt

$$W(r_{nn'}) = -W_0 + W_1 \cdot \sum_{\substack{n \\ (n<n')}} \sum_{n'} \frac{r_{nn'}^2}{a^2} \tag{4.27}$$

mit

$$W_0 = -\frac{3}{4\pi i} \int_0^\infty \mathrm{d}t \left\{ \log \frac{(1 + K_{0(+)}(t^2))^Z}{1 + Z \cdot K_{0(+)}(t^2)} - \log \frac{(1 + K_{0(-)}(t^2))^Z}{1 + Z \cdot K_{0(-)}(t^2)} \right\} \tag{4.27'}$$

und

$$W_1 = \frac{1}{2\pi} \operatorname{Im} \int_0^\infty \mathrm{d}t \, L_{(+)}(t^2) \,, \tag{4.27''}$$

wo $L(z)$ in (I.23′) definiert ist. W_0 ist die Bindungsenergie für «zusammenklebende» Elektronen; mit (4.8) und (I.2) kann man auch schreiben

$$W_0 = Z \cdot \left[E_0 - \frac{1}{Z} \cdot E_0 \Big|_{r_{c0} \to Z r_{c0}} \right]. \tag{4.28}$$

[27] H. SALECKER: *Zeits. f. Naturforsch.*, **10** *a*, 349 (1955).

Dies ergibt im Falle der r_{c0}-Abhängigkeit von (4.10)

$$\frac{W_0}{Z} = \left(1 - \frac{1}{\sqrt{Z}}\right) \cdot E_0 ,$$

d.h. die Kräfte haben für große Z Sättigungscharakter (W_0 proportional Z), ebenso wie in anderen Paar-Theorien mit starker Kopplung [1,2,24].

Für den weiter oben explizit behandelten Fall (I.13) erhält man

$$\left\{ \begin{aligned} \frac{\pi a}{3} W_0 &= Z \cdot \left[F_0\left(\frac{a}{r_{c0}}\right) - \frac{1}{Z} \cdot F_0\left(\frac{a}{Z r_{c0}}\right) \right] \\ \frac{\pi a}{4} W_1 &= \frac{1}{Z^2} \cdot F_1\left(\frac{a}{Z r_{c0}}\right) , \end{aligned} \right. \tag{4.29}$$

wo F_0 in (4.14) definiert und

$$F_1(\xi) = \operatorname{Im} \frac{1}{\xi^2} \cdot \int_0^\infty \frac{d\tau}{(1 - i\tau)^2} \cdot \frac{(1 - i\tau)^2 + \tau^2}{(1 - i\tau)^2 - (1/\xi)} \tag{4.30}$$

ist. Die Auswertung dieses Integrals gibt

$$F_1(\xi) = \frac{1}{\xi} \cdot \left\{ \sqrt{\xi} \operatorname{arctg} \frac{1}{\sqrt{\xi}} - 1 + \log \frac{1 + \xi}{\xi} \right\} ; \qquad \xi > 0 \tag{4.30'}$$

(hier bedeutet arctg den Hauptwert) und

$$F_1(\xi) = \frac{1}{|\xi|} \cdot \left\{ -\frac{\sqrt{\xi}}{2} \log \left| \frac{1 + \sqrt{|\xi|}}{1 - \sqrt{|\xi|}} \right| + 1 - \log \left| \frac{1 - |\xi|}{\xi} \right| \right\} ; \quad \xi < 0 . \tag{4.30''}$$

Der Verlauf von $F_1(\xi)$ ist in Fig. 3 wiedergegeben. Man sieht daraus, daß für $r_{c0} < 0$ die Kräfte an der Stelle $a/Z r_{c0} \simeq -0.36$ das Vorzeichen wechseln. Bei festem $r_c > 0$ folgt aber nach (4.24), daß dieser Vorzeichenwechsel nur für $Z = 2$ ($0.36 < 1/Z$) existiert, während für $Z \geqslant 3$ und $r_{c0} < 0$ die Kräfte stets abstoßend sind. Anderseits folgt durch Vergleich von (4.27, 29) mit (2.37), daß bei gegebenem $Z \geqslant 2$ ein Wert von $a/r_{c0} > 0$ existiert, bei dem die Konfiguration im Gleichgewicht ist. Die Bedingung dafür, $W_1 = \beta/3 \cdot E_{\text{coul}}$, ergibt mit (2.32) und (I.14)

$$F_1\left(\frac{a}{Z r_{c0}}\right) = \frac{\pi \cdot Z}{6 \cdot 137} .$$

5. – Theorie der Streuung.

Zur Behandlung der Streuung erweist sich die Hauptachsentransformation (3.1) als zu speziell. Dies ist der Grund dafür, daß in allen Formulierungen einer Paar-Theorie mit skalarer Kopplung, die auf (3.1) basieren, das Streuproblem keine befriedigende Darstellung erfahren hat. In dieser Feststellung liegt auch ein Hauptmotiv für die Arbeit von KLEIN und MCCORMICK [7] (s. dort die Einleitung), in welcher die Theorie der Streuung auf anderem Weg gelöst ist.

Statt (3.1) wählen wir als Ausgangspunkt die allgemeinste lineare Transformation der P, Q,

$$\begin{cases} P'^{+} = MP^{+} - iKQ \\ Q' \;\; = NQ + iLP^{+} . \end{cases} \tag{5.1}$$

Die Wahl der Matrizen M, N, K, L ist durch die Realitätsbedingungen (2.45), die V.R. (2.47) und die Forderung (3.4) derart eingeschränkt, daß gilt

$$\begin{cases} M^{-} = M^{T} ; \qquad N^{-} = N^{T} \\ K^{+} = -K^{T} ; \qquad L^{+} = -L^{T} \end{cases} \tag{5.2}$$

$$\begin{cases} MK^{+} + KM^{+} = 0 \\ NL^{+} \;\; = LN^{+} \;\; = 0 \\ MN^{+} \;\; + KL^{+} = 1 \end{cases} \tag{5.3}$$

$$\begin{cases} (a) \quad M^{+}M + L^{+}BL \;\; = 1 \\ (b) \quad K^{+}K \;\; + N^{+}BN = \Omega^{2} \\ (c) \quad M^{+}K \;\; + L^{+}BN \;\; = 0 . \end{cases} \tag{5.4}$$

Aus (5.4), Gleichung (b), folgt mit (2.40′, 44′, 46) und (5.2) wieder die Beziehung (3.6). Bildet man ferner aus (5.4) die Gleichung $(a)\cdot N^{+} - (c)\cdot L^{+}$, so folgt mit (5.3)

$$M = N \tag{5.5}$$

und ebenso aus $L\cdot(b) + N\cdot(c)$

$$K = L\Omega^{2} . \tag{5.6}$$

Weiter folgt aus $M\cdot(b) + K\cdot(c)$ mit Hilfe von (5.3, 5)

$$BM = M\Omega^{2} \tag{5.7}$$

and analog aus $(a)\cdot K^{+}+(c)\cdot M^{+}$ unter Benützung von (5.3, 6) und (2.46)

$$BK = K\Omega^2 . \tag{5.8}$$

Die Transformation von Abschnitt 3 ist in diesen Gleichungen als der Spezialfall $K=0$ enthalten; wir schreiben hier

$$M = M_{(1)} ; \quad (K = 0) . \tag{5.9}$$

Für $K\neq 0$ ist es zweckmäßig, die folgenden Matrizen einzuführen

$$\begin{cases} M = N = \dfrac{1}{2}(M_{(+)} + M_{(-)}) , \\ K\dfrac{1}{\Omega} = L\Omega = \dfrac{1}{2}(M_{(+)} - M_{(-)}) . \end{cases} \tag{5.10}$$

Zerlegen wir die Variablen P, Q nach positiven und negativen Frequenzen von $H'_{\mathrm{Ph}}(P,Q)$, gemäß (4.1),

$$\begin{cases} P^{+}_{(\pm)} = \mp i\left(\dfrac{\Omega}{2}\right)^{\frac{1}{2}}\cdot\begin{Bmatrix} a \\ a^{+r}\end{Bmatrix} = \dfrac{1}{2}\cdot(P^{+} \mp i\Omega Q) , \\ Q_{(\pm)} = (2\Omega)^{-\frac{1}{2}}\cdot\begin{Bmatrix} a \\ a^{+r}\end{Bmatrix} = \dfrac{1}{2}\cdot(Q \pm i\Omega^{-1}P^{+}) \end{cases} \tag{5.11}$$

so kann man für (5.1) schreiben

$$\begin{cases} P'^{+} = P'^{+}_{(-)} + P'^{+}_{(+)} ; \qquad P'^{+}_{(\pm)} = M_{(\pm)}P^{+}_{(\pm)} \\ Q' \;\; = Q'_{(+)} + Q'_{(-)} ; \qquad Q'_{(\pm)} = M_{(\pm)}Q_{(\pm)} . \end{cases} \tag{5.12}$$

Ferner gibt (5.7, 8) mit (5.10)

$$BM_{(\pm)} = M_{(\pm)}\Omega^2 . \tag{5.13}$$

Weiter bilden wir aus den Gleichungen (5.4) die drei Ausdrücke $\Omega^{+}\cdot(a)\pm$ $\pm\Omega^{+}\cdot(c)\cdot(1/\Omega)\pm(c)^{+}+(b)\cdot(1/\Omega)$, $\Omega^{+}\cdot(a)-\Omega^{+}\cdot(c)\cdot(1/\Omega)+(c)^{+}-(b)\cdot(1/\Omega)$ und berücksichtigen (5.10, 13). Mit den Abkürzungen

$$M^{+}_{(\pm)}M_{(\pm)} = e_{(\pm)} \tag{5.14}$$

$$M^{+}_{(-)}M_{(+)} = s \tag{5.15}$$

erhält man dann

$$\Omega^{+}e_{(\pm)} + e_{(\pm)}\Omega = \Omega^{+} + \Omega \tag{5.16}$$

$$\Omega^{+}s - s\Omega = \Omega^{+} - \Omega , \tag{5.17}$$

oder in Komponenten

$$(\Omega_k^+ + \Omega_{k'}) \cdot (e_{(\pm)kk'} - \delta_{kk'}) = 0$$
$$(\Omega_k^+ - \Omega_{k'}) \cdot (s_{kk'} - \delta_{kk'}) = 0 .$$

Bisher waren in diesem Abschnitt die Eigenwerte $\Omega_0^2 < 0$ in der Diagonalmatrix Ω^2 nicht ausgeschlossen worden, sodaß zwar (3.6) richtig ist, aber nicht notwendig $\Omega^+ = \Omega$. Vielmehr gilt

$$\text{(5.18)} \qquad \begin{cases} \Omega_k - \Omega_k^+ = +|\Omega_k| \\ \Omega_0 - \Omega_k^+ = +i|\Omega_0| \end{cases}$$

(die rechte Seite ist eine Festsetzung des Vorzeichens), sodaß aus der Komponenten-Form von (5.16, 17) folgt

$$\text{(5.19)} \qquad \begin{cases} e_{(\pm)k,k'} = \delta_{kk'} \\ e_{(\pm)0,k} + e_{(\pm)k,0} = 0 \\ e_{(\pm)0,0} \text{ bel. diagonal } (^{20}) \end{cases}$$

und

$$\text{(5.20)} \qquad \begin{cases} s_{k,k'} = \begin{cases} 0 ; & \Omega_k \neq \Omega_{k'} \\ \text{bel. auf Energieschale } & \Omega_k = \Omega_{k'} \end{cases} \\ s_{0,k} = s_{k,0} = 0 \\ s_{0,0} = 1 \quad (^{20}) . \end{cases}$$

Aus (5.19) ist ersichtlich, daß man in jedem Fall die $e_{(\pm)} = 1$ wählen kann. Die Frage ist nur, unter was für Bedingungen dann gleichzeitig auch die Vollständigkeitsrelationen

$$\text{(5.21)} \qquad M_{(\pm)} M_{(\pm)}^+ = 1$$

erfüllt sein können. Um dies zu untersuchen setzen wir (5.10) in die 3. Gleichung (5.3) ein und erhalten

$$\text{(5.22)} \qquad M_{(+)}(1 + \Omega/\Omega^+) M_{(+)}^+ + M_{(-)}(1 + \Omega/\Omega^+) M_{(-)}^+ +$$
$$+ M_{(+)}(1 - \Omega/\Omega^+) M_{(-)}^+ + M_{(-)}(1 - \Omega/\Omega^+) M_{(+)}^+ = 4 ,$$

oder in Komponenten, mit Hilfe von (5.18),

$$\sum_{k'' \neq 0} \tfrac{1}{2}(M_{(+)kk''} M_{(+)k''k'}^+ + M_{(-)kk''} M_{(-)k''k'}^+) + \tfrac{1}{2}(M_{(+)k,0} M_{(-)0,k'}^+ + M_{(-)k,0} M_{(+)0,k'}^+) = \delta_{kk'} .$$

Mit (5.21) folgt daraus

$$(M_{(+)k,0} - M_{(-)k,0})(M^{+}_{(+)0,k'} - M^{+}_{(-)0,k'}) = 0 ,$$

oder als notwendige Bedingung für die Vollständigkeit der $M_{(\pm)}$

$$M_{(+)k,0} = M_{(-)k,0} .$$

Von nun an schließen wir die Eigenwerte $\Omega_0^2 < 0$ wieder aus. Es ist dann zweckmäßig, zum kontinuierlichen k-Raum überzugehen, sodaß gilt

$$\Omega = \Omega^{+} = \omega . \tag{5.23}$$

Jetzt wird nämlich gemäß (3.4)

$$H'_{\mathrm{Ph}}(P, Q) = H_0(P, Q) . \tag{5.23'}$$

Die P, Q sind also freie Felder und (5.12) kann aufgefaßt werden als Transformation von einfallenden Feldern

$$P = P^{\mathrm{ein}} ; \qquad Q = Q^{\mathrm{ein}} \tag{5.24}$$

auf wechselwirkende Felder P', Q'. Der Operator Λ der kanonischen Transformation

$$\begin{cases} P'_k = \Lambda^{-1} P_k \Lambda \\ Q'_k = \Lambda^{-1} Q_k \Lambda \\ H'_{\mathrm{Ph}}(P, Q) = \Lambda^{-1} H_{\mathrm{Ph}}(P, Q) \Lambda \end{cases} \tag{5.25}$$

hat dann (aber nur dann) die Bedeutung der Møller'schen Streumatrix und ließe sich mit Hilfe von (5.12) explizit konstruieren (vergl. Klein un McCormick (7)), eine andere Darstellung von Λ geben wir in Anhang II). Entsprechend (5.25) lautet die Transformation der Vacuum-Zustände (4.3) bzw. der n-Photonen Eigenzustände von $H_0(P', Q')$ und $H'_{\mathrm{Ph}}(P, Q) = H_0(P, Q)$

$$\begin{cases} |0\rangle = \Lambda |0'\rangle \\ |1_{k_1} \dots 1_{k_n}\rangle = \Lambda |1'_{k_1} \dots 1'_{k_n}\rangle . \end{cases} \tag{5.25'}$$

Die Gleichungen (5.17, 22) lauten jetzt

$$\omega s - s\omega = 0 \tag{5.17'}$$

$$\tfrac{1}{2}(M_{(+)} M^{+}_{(+)} + M_{(-)} M^{+}_{(-)}) = 1 \tag{5.22'}$$

und die Willkür in der Wahl von $e_{(\pm)}$, (5.19), fällt weg, sodaß streng gilt

$$M^{+}_{(\pm)} M_{(\pm)} = 1 \,. \tag{5.21'}$$

Damit folgt durch Bildung von $M^{+}_{(\pm)} \cdot (5.22') \cdot M_{(\pm)}$ für die Matrix (5.15)

$$s s^{+} = s^{+} s = 1 \,. \tag{5.26}$$

Wir vervollständigen die Reihe der Bedingungen, denen die Matrizen $M_{(\pm)}$ im Falle (5.23) unterworfen sind, durch Hinzunahme von (5.2) und der ersten beiden Gleichungen (5.3), welche mit Benützung von (5.10) ergeben

$$M_{(-)} = M^{T}_{(+)} \tag{5.27}$$

und

$$\begin{cases} M_{(+)} \omega M^{+}_{(+)} = M_{(-)} \omega M^{+}_{(-)} \,, \\ M_{(+)} \dfrac{1}{\omega} M^{+}_{(+)} = M_{(-)} \dfrac{1}{\omega} M^{+}_{(-)} \,. \end{cases} \tag{5.28}$$

Endlich zeigen wir noch, daß jetzt auch die Vollständigkeitsrelationen (5.21) streng folgen. Zunächst erhält man aus der 1. Gleichung (5.28) durch Multiplikation von rechts mit $M_{(+)}$ unter Benützung von (5.17', 21')

$$M_{(+)} = M_{(-)} \cdot s \,. \tag{5.29}$$

Dies eingesetzt in (5.22') gibt mit (5.26) in der Tat (5.21).

Es sei daran erinnert, daß in keinem der obigen Schritte die Existenz der Inversen einer Matrix vorausgesetzt worden ist. Es ist ferner leicht zu zeigen, daß umgekehrt die Beziehungen (5.26, 28, 29) aus (5.21, 21', 17') folgen.

Wir gehen nun zur expliziten Konstruktion der Transformationsmatrizen $M_{(\pm)}$ und $M_{(1)}$ für den Fall (5.23). Die Bestimmungsgleichungen (5.7, 13) lauten ausgeschrieben, mit (2.46),

$$(\omega_k^2 - \omega_{k'}^2) M_{(\,)kk'} = - g_0 \sum_{k''} A_{kk''} M_{(\,)k''k'} \,. \tag{5.30}$$

Nun führen wir «Green'sche Funktionen» $G_{(\,)kk'}$ ein, welche der Gleichung genügen

$$(\omega_k^2 - \omega_{k'}^2) G_{(\,)kk'} = + 1 \,, \tag{5.31}$$

oder explizit

$$G_{(\pm)kk'} = \frac{1}{\omega_k^2 - (\omega_{k'}^2 \pm i\varepsilon)}, \tag{5.32}$$

$$G_{(1)kk'} = \frac{1}{2}(G_{(+)kk'} + G_{(-)kk'}) = \frac{P}{\omega_k^2 - \omega_{k'}^2}, \tag{5.33}$$

wobei im letzten Ausdruck unter P der Hauptwert in einer ω_k^2-oder $\omega_{k'}^2$-Integration gemeint ist. Damit kann man (5.30) in der Form einer Integralgleichung schreiben

$$M_{(\,)kk'} = m_{(\,)kk'} - G_{(\,)kk'} \cdot g_0 \sum_{k''} A_{kk''} M_{(\,)k''k'}. \tag{5.34}$$

Hier ist $m_{(\,)kk'}$ eine Lösung der wechselwirkungsfreien Gleichung

$$(\omega_k^2 - \omega_{k'}^2) m_{(\,)kk'} = 0 \tag{5.35}$$

und ist somit identisch null außerhalb der Energieschale $\omega_k = \omega_{k'}$. Die Auflösung von (5.34) bietet keine Schwierigkeiten; dies ist dem Umstand zuzuschreiben, daß der Kern $A_{kk'}$ gemäß (2.43) ausgeartet ist. Durch Bildung von $\sum_k \alpha_k \cdot$(5.34) folgt zunächst

$$\sum_{k'} \alpha_{k'} M_{(\,)k'k} = (1 + C_{(\,)k})^{-1} \sum_{k'} \alpha_{k'} m_{(\,)k'k}, \tag{5.36}$$

wobei

$$C_{(\,)k} \equiv C_{(\,)}(\omega_k^2) \equiv g_0 \sum_{k'} \alpha_{k'} \times \alpha_{k'}^* G_{(\,)k'k} \tag{5.37}$$

die Matrix (2.53) mit Komponenten (2.53′), in offensichtlicher Modifikation, ist. (5.36) eingesetzt in (5.34) gibt

$$M_{(\,)kk'} = \sum_{k''} \{\delta_{kk''} - g_0 \alpha_k^* (1 + C_{(\,)k'})^{-1} \alpha_{k'} G_{(\,)kk'}\} m_{(\,)k''k'}. \tag{5.38}$$

Dies läßt sich noch einfacher schreiben, wenn man berücksichtigt, daß $m_{(\,)k''k'} = 0$ ist für $\omega_{k''}^2 - \omega_{k'}^2 \neq 0$, und infolgedessen $C_{(\,)k'}$ durch $C_{(\,)k''}$ und $G_{(\,)kk'}$ durch $G_{(\,)kk''}$ ersetzt werden darf:

$$M_{(\,)} = (1 - R_{(\,)}) \cdot m_{(\,)}, \tag{5.38′}$$

mit

$$R_{(\,)kk'} = g_0 \alpha_k^* (1 + C_{(\,)k'})^{-1} \alpha_{k'} G_{(\,)kk'}. \tag{5.39}$$

In Anhang III werden für den Fall $M_{(\pm)}$ die folgenden Relationen abgeleitet

$$R_{(\pm)}R^{+}_{(\pm)} - R_{(\pm)} - R^{+}_{(\pm)} = 0 \tag{5.40}$$

$$R^{+}_{(\pm)}R_{(\pm)} - R_{(\pm)} - R^{+}_{(\pm)} = 0 \tag{5.40'}$$

$$R^{+}_{(\pm)} = R^{x}_{(\mp)}\,, \tag{5.41}$$

die erste allerdings nur unter der Voraussetzung, daß $\Delta(z)$ keine Wurzeln außerhalb der positiv reellen Achse hat (vergl. KLEIN und MCCORMICK [7], Appendix A), ferner

$$(1 - R^{+}_{(-)})(1 - R_{(+)}) = \delta_{kk'} - \overline{R}_{kk'}\,\delta(\omega_k - \omega_{k'})\,, \tag{5.42}$$

mit

$$\overline{R}_{kk'} \equiv i\,\frac{\pi}{\omega_k}\cdot g_0\alpha^{*}_{k}\,(1 + C_{(+)k})^{-1}\alpha_{k'}\,. \tag{5.42'}$$

Aus diesen Beziehungen folgt, daß man mit

$$m_{(\pm)} = 1 \tag{5.43}$$

die Bedingungen (5.21, 21', 27, 17') erfüllen kann. Physikalisch bedeutet (5.43), daß die Felder bei Ausschalten der Wechselwirkung in P^{ein}, Q^{ein}, (5.24), übergehen; man hat eine Darstellung durch einlaufende und retardierte Felder.

Wesentlich schwieriger ist der Fall $M_{(1)}$, für den in Anhang III folgende Beziehungen abgeleitet sind

$$R^{+}_{(1)} = R^{x}_{(1)}\,, \tag{5.44}$$

$$R^{+}_{(1)}R_{(1)} - R_{(1)} - R^{+}_{(1)} = r^{2}_{(1)}\,, \tag{5.45}$$

mit

$$r_{(1)kk'} = (r^{+}_{(1)})_{kk'} = \pi\, g_0\alpha^{*}_{k}(1 + C_{(1)k})^{-1}\alpha_{k'}\,\delta(\omega^{2}_{k} - \omega^{2}_{k'})\,. \tag{5.45'}$$

Es ist hier also keineswegs möglich, die Bedingungen (3.3, 2, 2') mit $m_{(1)} = 1$ zu befriedigen ([28]). Man könnte nun daran denken, $m_{(1)}$ so zu wählen, daß es ebenfalls (3.3, 2, 2') erfüllt, was bedeuten würde, daß auch hier das Ab-

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([28]) Dies wurde noch von MORPURGO und TOUSCHEK [6] angenommen. KLEIN und MCCORMICK [7] haben dann in ihrem Fall für $m_{(1)} = 1$ gezeigt (Appendix A), daß die Unitarität von $M_{(1)}$, (3.2'), verletzt ist.

schalten der Wechselwirkung einen Sinn hat. Dann müßte aber gelten

$$r_{(1)}m_{(1)} = 0$$

und da $m_{(1)}$ wegen (5.35) von der Form ist

$$m_{(1)kk'} = \mu_{kk'}\delta(\omega_k^2 - \omega_{k'}^2) ,$$

würde daraus weiter folgen

$$R_{(1)}m_{(1)} = 0 ; \qquad M_{(1)} = m_{(1)} ; \qquad AM_{(1)} = 0 ,$$

also ein absurdes Resultat. Man schließt daraus, daß im Fall $M_{(1)}$ das Abschalten der Wechselwirkung physikalisch nicht sinnvoll ist; man hat eine Darstellung durch stehende Wellen.

Wir lassen die Frage offen, wie die Matrix $m_{(1)}$ zu wählen ist, um die Vollständigkeitsrelation (3.2) zu erfüllen, und gehen über zur Bestimmung der S-Matrix aus $M_{(\pm)}$. Zu diesem Zweck haben wir analog zu (5.12, 24) auslaufende Felder einzuführen durch

$$\text{(5.46)} \qquad \begin{cases} P'_{(\pm)} = M_{(\mp)}P^{\pm\,\text{aus}}_{(\pm)} \\ Q'_{(\pm)} = M_{(\mp)}Q^{\text{aus}}_{(\pm)} . \end{cases}$$

Wir bemerken dazu, daß die Vertauschung von $M_{(+)}$ und $M_{(-)}$ gegenüber (5.12) wegen (5.27) gerade der Zeitumkehr ([16]) entspricht. Die S-Matrix vermittelt dann die kanonische Transformation

$$\text{(5.47)} \qquad \begin{cases} P_k^{\text{aus}} = S^{-1}P_k^{\text{ein}}S \\ Q_k^{\text{aus}} = S^{-1}Q_k^{\text{ein}}S \\ H_0(P^{\text{aus}}, Q^{\text{aus}}) = S^{-1}H_0(P^{\text{ein}}, Q^{\text{ein}})S . \end{cases}$$

Durch Gleichsetzen der positiven (und negativen) Frequenzteile von (5.12) und (5.46) folgt mit (5.24, 47)

$$\text{(5.47')} \qquad Q'_{(+)} \equiv M_{(+)}Q_{(+)} = M_{(-)}S^{-1}Q_{(+)}S ,$$

oder nach Multiplikation von links mit $M^{-}_{(-)}$ und Berücksichtigung von (5.11, 23) und (5.21', 15, 17')

$$\text{(5.48)} \qquad S^{-1}a_kS = b_k \equiv \sum_{k'} s_{kk'}a_{k'} ,$$

wobei s jetzt durch (5.42) gegeben ist. Daraus ließe sich S explizit konstruieren (7) (eine andere Darstellung findet sich in Anhang II). Hier interessieren wir uns insbesondere für die 1-Photon Untermatrix. Mit den physikalischen Zuständen (5.25′) folgt aus (5.48, 42)

$$\langle 1_k | S | 1_{k'} \rangle / \langle 0 | S | 0 \rangle = s_{kk'} = \delta_{kk'} - \overline{R}_{kk'} \delta(\omega_k - \omega_{k'}) . \tag{5.49}$$

Unter der Voraussetzung

$$\langle 0 | S | 0 \rangle = 1 , \tag{5.50}$$

auf die wir im nächsten Abschnitt zurückkommen, wird der Wirkungsquerschnitt

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{V^2}{(2\pi)^4} \cdot \omega_k^2 | \overline{R}_{kk'} |^2_{\omega_{k'} = \omega_k} , \tag{5.51}$$

oder mit (5.42′), (2.60)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{3}{2} r_0 \right)^2 \cdot \alpha_k^* (1 + C_{(+)k}) \, {}^1\alpha_{k'} |^2_{\omega_{k'} = \omega_k} . \tag{5.51′}$$

6. – Renormalisationsfragen. Schluß.

Nachdem wir in Abschnitt 4 zwei Renormierungsvorschriften diskutiert haben, aus denen übereinstimmend die Nichtexistenz einer Ladungsrenormalisation folgte, zeigen wir jetzt, daß man dasselbe Resultat auch aus dem Wirkungsquerschnitt (5.51′) erhält. Als Vorschrift für die Renormalisation der Kopplungskonstanten verlangen wir nach Thirring (9), daß (5.51′) für isolierte Elektronen (alle $r_{nn'} \to \infty$) und Photonen kleiner Energie ($\omega_k \to 0$) in die störungstheoretische Formel mit renormierter Kopplungskonstante g übergeht. Zusammen mit der Masserenormalisation (2.59′) folgt dann wieder die obige Behauptung.

Das Matrixelement für den Übergang zwischen freien 1-Photon Zuständen lautet mit g anstelle von g_0

$$H_{kk'} \equiv \langle 1'_k | H_1(P', Q') | 1'_{k'} \rangle = \tfrac{1}{2} g \{ A_{kk'} + \tfrac{1}{2} \operatorname{Spur} (A \omega^{-1}) \delta_{kk'} \} . \tag{6.1}$$

Damit wird der renormierte störungstheoretische Wirkungsquerschnitt

$$\frac{\mathrm{d}\sigma^{r.s.}}{\mathrm{d}\Omega} = \left(\frac{V}{2\pi} \right)^2 \omega_k^2 | H_{kk'} |^2_{\substack{\omega_k = \omega_{k'} \\ (k \neq k')}} , \tag{6.2}$$

oder mit der analogen Definition zu (2.60), nämlich

$$r_c = g \cdot \frac{\Gamma}{6\pi}, \tag{6.3}$$

$$\frac{\mathrm{d}\sigma^{r.s.}}{\mathrm{d}\Omega} = \left(\frac{3}{2} r_c\right)^2 |A_{kk'}|^2_{\omega_k = \omega_{k'}}. \tag{6.2'}$$

Andrerseits folgt aus (5.51') für den Fall, daß alle Elektronenabstände unendlich sind und unter Berücksichtigung von (2.57, 43)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{3}{2} r_{c0}\right)^2 |A_{kk'}/D_{0(+)}(\omega_k^2)|^2_{\omega_k = \omega_{k'}}. \tag{6.4}$$

Durch Vergleich von (6.2') und (6.4) erhält man mit (2.59') im Grenzfall $\omega_k \to 0$

$$\frac{1}{r_c^2} = |D_{0(+)}(0)|^2 \cdot \frac{1}{r_{c0}^2} = D_0^2(0)/r_{c0}^2 = \left(\frac{1}{r_{c0}} + \frac{1}{a}\right)^2, \tag{6.5}$$

also wieder (4.24) ([29]).

Die Behauptung der Nichtexistenz einer Ladungsrenormalisation basiert hier allerdings wesentlich auf der Voraussetzung (5.50). Wäre diese nicht erfüllt, so würde die obige Vorschrift eine « Ladungsrenormalisation zweiter Art » ([7]) implizieren, indem die Matrix $\langle 1_k | S | 1_{k'} \rangle$ nur relativ zu $\langle 0 | S | 0 \rangle$ definiert wäre. Nun kann man aber (5.50) unter der Voraussetzung der Unitarität von S, welche aus der Darstellung (II.19) unmittelbar hervorgeht, beweisen. Zunächst folgt (für $g_0 > 0$) wegen der Vollständigkeit der Eigenzustände $|1_{k_1} \dots 1_{k_n}\rangle$ von $H'_{\mathrm{Ph}}(P, Q)$

$$\langle 0 | 0 \rangle = \langle 0 | S^+ S | 0 \rangle = |\langle 0 | S | 0 \rangle|^2 + \sum_{n=1}^{\infty} \sum_{k_1} \dots \sum_{k_n} |\langle 1_{k_1} \dots 1_{k_n} | S | 0 \rangle|^2 . \tag{6.6}$$

Ferner erhält man aus (5.48) durch Induktion

$$a_{k_1} \dots a_{k_n} S = S b_{k_1} \dots b_{k_n}, \tag{6.7}$$

sodaß gilt

$$\langle 1_{k_1} \dots 1_{k_n} | S | 0 \rangle = (n!)^{-\frac{1}{2}} \langle 0 | a_{k_1} \dots a_{k_n} S | 0 \rangle = (n!)^{-\frac{1}{2}} \langle 0 | S b_{k_1} \dots b_{k_n} | 0 \rangle = 0, \tag{6.8}$$

([29]) Es ist eine bekannte Tatsache, daß in der Quantenelektrodynamik alle hier diskutierten Renormierungsvorschriften denselben Wert für die Ladung liefern. Vergl. z.B. die Einleitung bei Deser, Thirring und Goldberger ([13]).

letzteres da die b_k gemäß ihrer Definition (5.48) nur Absorptionsoperatoren a_k enthalten. Man hat also

$$|\langle 0|S|0\rangle|^2 = \langle 0|0\rangle\,. \tag{6.9}$$

Jetzt könnte man noch daran denken, das Vacuum zu renormieren gemäß

$$\langle 0|0\rangle = N^2\,. \tag{6.10}$$

Damit würde aber (4.4)

$$\langle H_{\mathrm{Ph}}\rangle_0 = \tfrac{1}{2}\sum_k (N^2\Omega_k - \omega_k)\,, \tag{6.11}$$

was natürlich fur $N \neq 1$ divergiert ([30]). Mit $N = 1$ folgt aber die Behauptung (5.50), wenn man bedenkt, daß S durch die Gleichung (5.48) und die Unitarität nur bis auf einen c-Zahl Faktor vom Betrage eins bestimmt ist.

Im Hinblick darauf, daß sowohl in der Arbeit von KLEIN und MCCORMICK ([7]) wie auch in derjenigen von MORPURGO und TOUSCHEK ([6]) die Matrix Λ benutzt wurde, ist es noch von Interesse, auch zu zeigen, daß

$$\langle 0|\Lambda|0\rangle = 1 \tag{6.12}$$

gilt. Dies läßt sich aber nicht analog zu obigem Beweis machen. Λ ist zwar durch eine zu (5.48) analoge Gleichung bestimmt, nämlich

$$\Lambda^{-1}a_k\Lambda = a'_k\,, \tag{6.13}$$

wobei für die a'_k aus (5.11, 12, 23, 25) folgt

$$\begin{aligned} a'_k &= \left(\frac{\omega_k}{2}\right)^{\frac{1}{2}} Q'_k + i(2\omega_k)^{-\frac{1}{2}} P'_k = \\ &= \sum_{k'} \{(\omega/2)^{\frac{1}{2}} M_{(+)}(2\omega)^{-\frac{1}{2}} + (2\omega)^{-\frac{1}{2}} M_{(+)}(\omega/2)^{\frac{1}{2}}\}_{kk'} \cdot a_{k'} + \\ &+ \sum_{k'} \{(\omega/2)^{\frac{1}{2}} M_{(-)}(2\omega)^{-\frac{1}{2}} - (2\omega)^{-\frac{1}{2}} M_{(-)}(\omega/2)^{\frac{1}{2}}\}_{kk'} \cdot a^+_{k'}\,. \end{aligned} \tag{6.13'}$$

Doch enthält a'_k Unterschied zu b_k auch Emissionsoperatoren a^+_k. Es gibt aber

([30]) Für $Z = 1$ bestünde zwar die Möglichkeit, die Selbstenergie weg zu renormieren; durch

$$N^2 = \lim_{k\to\infty} \sum_{\substack{k \\ \omega_k < K}} \omega_k \Big/ \sum_{\substack{k \\ \omega_k < K}} \Omega_k$$

bringt man nämlich (6.11) zum Verschwinden.

noch einen andern Weg. Aus der Darstellung (II.17) bzw. (II.19) folgt nämlich mit (4.3), (5.23)

$$\langle 0|\Lambda|0\rangle = \lim_{T\to\infty} \exp\left[-i\sum_k \omega_k \cdot \frac{T}{2}\right] \langle 0|\exp[iH_0(P',Q')\cdot T]|0\rangle\,, \tag{6.14}$$

bzw.

$$\langle 0|S|0\rangle = \lim_{T\to\infty} \exp\left[-i\sum_k \omega_k \cdot T\right] \langle 0|\exp[iH_0(P',Q')\cdot 2T]|0\rangle\,, \tag{6.15}$$

also durch Vergleich

$$\langle 0|\Lambda|0\rangle = \langle 0|S|0\rangle\,. \tag{6.16}$$

Damit ist der Beweis, daß im Falle $g_0 > 0$ auch aus dem Wirkungsquerschnitt keine Ladungsrenormalisation folgt, vollständig. Nun hatten wir in Abschnitt 4 festgestellt, daß der renormierte Ausdruck (4.19′) für die Korrektur zum Coulomb'schen Gesetz auch für $g_0 < 0$, insbesondere beim Übergang zu Punktelektronen ($a \to 0$), sinnvoll bleibt. Dasselbe gilt auch für den renormierten Wirkungsquerschnitt. Wir zeigen nämlich jetzt noch, daß die renormierte 1-Photon S-Matrix für ein Punktelektron ($Z = 1$, $a \to 0$) in diejenige der klassischen Theorie, (IV.17′), übergeht.

Für $Z = 1$ lautet (5.49, 42′)

$$s_{kk'} = \delta_{kk'} - i\,\frac{\pi}{\omega_k}\,\frac{6\pi r_{c0}}{V}\left(A_{kk'}/D_{0(+)}(\omega_k^2)\right)\cdot\delta(\omega_k - \omega_{k'})\,, \tag{6.17}$$

wobei aus (2.43) folgt

$$A_{kk'} = (\boldsymbol{e}_k\boldsymbol{e}_{k'})v_{\boldsymbol{k}}^* v_{\boldsymbol{k}'}\,. \tag{6.17′}$$

Definieren wir eine renormierte Funktion $D(z)$ durch

$$D(z) \equiv \frac{D_0(z)}{D_0(0)}\,, \tag{6.18}$$

so wird mit (4.24), (2.58′, 25′)

$$D(z) = \frac{r_c}{r_{c0}}\cdot D_0(z) = 1 - \frac{r_c}{a} + r_c\,\frac{2}{\pi}\int_0^\infty \frac{t^2 u(t)\,\mathrm{d}t}{t^2 \quad z} = 1 + z\cdot r_c\,\frac{2}{\pi}\int_0^\infty \frac{u(t)\,\mathrm{d}t}{t^2 - z} \tag{6.18′}$$

und analog zu (4.9)

$$D_{(\pm)}(t^2) = D_1(t^2) \pm i r_c\cdot|t|\cdot u(|t|) \tag{6.19}$$

mit

$$D_{(1)}(t^2) \equiv \frac{1}{2}\,(D_{(+)} + D_{(-)}) = 1 + t^2 \cdot r_c\,\frac{2}{\pi}\,\mathop{P}\int_0^\infty \frac{u(t')\,\mathrm{d}t'}{t'^2 - t^2}\,. \tag{6.19'}$$

Im limes $a \to 0$ ist nun

$$v_k = 1\,; \qquad u(t) = 1 \tag{6.20}$$

und

$$D_{(1)}(t^2) = 1\,, \tag{6.20'}$$

sodaß man für (6.17, 17′) erhält

$$s_{kk'} = \delta_{kk'} - i\,\frac{\pi}{\omega_k}\cdot\frac{6\pi r_c}{V}\cdot\frac{(\boldsymbol{e}_k\boldsymbol{e}_{k'})}{1 + i\omega_k r_c}\cdot\delta(\omega_k - \omega_{k'})\,, \tag{6.21}$$

was identisch ist mit (IV.17′).

Von besonderem Interesse ist hier der Pol

$$\omega_k = +\,i/r_c\,, \tag{6.22}$$

welchem in der klassischen Theorie die Lösung

$$\ddot{\boldsymbol{q}} \sim \exp\,[-\,i\omega_k t] = \exp\,[+\,t/r_c] \tag{6.23}$$

der feldfreien Gleichung zu (IV.3), d.h. die bekannte Dirac'sche Selbstbeschleunigung ([31]) entspricht. In der hier dargestellten Theorie, wie auch schon bei VAN KAMPEN ([1]), ist der Pol (6.22) identisch mit dem Eigenwert Ω_0 für $Z = 1$, $a = 0$. Denn mit (6.20) geht (6.18′) über in

$$D(-\,t^2) = 1 - |t|\,r_c\,. \tag{6.24}$$

Vor dem Grenzübergang $a \to 0$ stimmen aber gemäß der Definition (6.18) die Nullstellen von $D(z)$ mit denen von $D_0(z)$ überein (der Grenzübergang $a \to 0$ läßt sich natürlich für $D_0(z)$ nicht durchführen, sodaß dieser Fall aus Fig. 1 nicht ersichtlich ist).

Diese Bedeutung der Ω_0 als akausale Pole ($\mathrm{Im}\,\Omega_0 > 0$) der S-Matrix ist natürlich allgemein; denn die Matrix $(1 + C_{(+)}(\omega_k^2))^{-1}$ in (5.49, 42′) hat gerade die Singularitäten $\omega_k^2 = \Omega_0^2$, $\mathrm{Im}\,\omega_k > 0$. Interessant ist auch, daß das Auf-

([31]) P. A. M. DIRAC: *Proc. Roy. Soc.*, A **167**, 148 (1938). Vergl. auch K. WILDERMUTH: *Zeits. f. Naturforsch.*, **10***a*, 450 (1955).

treten dieser Pole ($g_0 < 0$) zwar die Vollständigkeitsrelationen (5.21) zerstört (dies ist eine natürliche Folge des Weglassens der Variablen P_0, Q_0), jedoch nicht die Unitarität der 1-Photon S-Matrix (5.49), wie in Anhang III gezeigt wird.

Die Situation bei der Renormalisation der Photonpaar-Theorie läßt sich also wie folgt zusammenfassen: Die Gültigkeit der klassischen Bewegungsgleichung für ein einzelnes Elektron, $\boldsymbol{p} = m\boldsymbol{r}$, verlangt eine Masserenormalisation. Mit dieser zusammen führen alle Renormierungsvorschriften für die Ladung bzw. die Kopplungskonstante im Falle $g_0 > 0$ auf die Nichtexistenz einer Ladungsrenormalisation, in Übereinstimmung mit der Abwesenheit einer Vacuumpolarisation in dieser Theorie. Für $g_0 < 0$ hat man zwar die Schwierigkeit der Eigenwerte Ω_0 und, als Folge davon, der Vacuum-Definition. Doch bleiben die meßbaren Folgerungen der Theorie, nämlich die renormierte Korrektur zum Coulomb'schen Gesetz und der renormierte Wirkungsquerschnitt sinnvoll, wenn man, wie es bei der Herleitung dieser Aussagen geschehen ist, den Beitrag der pathologischen Eigenwerte Ω_0 zur Hamiltonfunktion (3.4) einfach wegläßt (vergl. dazu den § 6 bei VAN KAMPEN [4]).

Ein weiterer Zug der Theorie ist, daß die Masserenormalisation wie in der klassischen Theorie des Elektrons, mit der Coulomb-Selbsenergie zusammenhängt und nicht, wie in der Quantenelektrodynamik, mit derjenigen, welche die Kopplung an das Photonen-Feld hervorruft.

* * *

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ANHANG I.

Auswertung von $C(z)$ und $\Delta(z)$.

Mit Hilfe von (2.13, 13′) und (2.28, 39, 40, 42) wird das Matrixelement (2.53′)

$$C_{\nu,\nu'}(z) = g_0 \sum_{\boldsymbol{k}} \frac{|v_{k_j}|^2}{|\boldsymbol{k}|^2 - z} \cdot \left(\delta_{jj'} - \frac{k_j k_{j'}}{|\boldsymbol{k}|^2}\right) \exp\left[i\boldsymbol{k}\boldsymbol{r}_{nn'}\right]. \tag{I.1}$$

Für $n = n'$ erhält man nach Ausführung der Winkelintegration

$$C_{jn,j'n}(z) = K_0(z)\,\delta_{jj'}\,, \tag{I.1'}$$

mit

$$K_0(z) \equiv D_0(z) - 1 = \frac{2}{3} g_0 \sum_{\boldsymbol{k}} \frac{|v_k|^2}{|\boldsymbol{k}|^2 - z}\,. \tag{I.2}$$

Die Auswertung der Elemente mit $n \neq n'$ ist etwas komplizierter. Zunächst führen wir im 3-dimensionalen Raum für jedes Indexpaar n, n' individuelle Koordinaten ein, die dadurch festgelegt sind, daß der Vektor $\boldsymbol{r}_{nn'}$ $(n > n')$ die Komponenten $(0, 0, r_{nn'})$ hat, während jene im festen System $x_j^{nn'}$ waren. Ferner soll $\boldsymbol{k}$ die alten bzw. neuen Komponenten k_j bzw. k_j' haben. Dann gibt es eine Transformation

$$\begin{cases} x_j^{nn'} = S_{j3}^{nn'} r_{nn'}\,, \\ k_j = \sum\limits_{j'} S_{jj'}^{nn'} k_{j'}'\,, \end{cases} \tag{I.3}$$

$$\sum_{j''} S_{jj''}^{nn'} S_{j'j''}^{nn'} = \sum_{j''} S_{j''j}^{nn'} S_{j''j'}^{nn'} = \delta_{jj'}\,. \tag{I.3'}$$

Für $n < n'$ setzen wir sie fest durch

$$S_{jj'}^{n'n} = S_{jj'}^{nn'}\,. \tag{I.3''}$$

Unter der unwesentlichen Annahme einer kugelsymmetrischen Abschneidefunktion v_k ist der winkelabhängige Teil von (I.1) im gestrichenen Koordinatensystem

$$\left(\delta_{jj'} - \frac{k_j' k_{j'}'}{|\boldsymbol{k}|^2}\right) \exp\left[\pm i k_3' r_{nn'}\right],$$

wobei das obere Vorzeichen für $n > n'$, das untere für $n < n'$ gilt. Die Mittelung dieses Ausdrucks über die Winkel gibt

$$\frac{1}{4\pi} \oint \mathrm{d}\Omega_{\boldsymbol{k}} \left(\delta_{jj'} - \frac{k_j' k_{j'}'}{|\boldsymbol{k}|^2}\right) \exp\left[\pm i k_3' r_{nn'}\right] = f_j(|\boldsymbol{k}| r_{nn'})\,\delta_{jj'}\,, \tag{I.4}$$

mit

$$\begin{cases} f_1(x) \equiv f_2(x) \equiv \dfrac{\sin x}{x} + \dfrac{\cos x}{x^2} - \dfrac{\sin x}{x^3}\,, \\ f_3(x) \equiv 2 - \dfrac{\cos x}{x^2} + 2\dfrac{\sin x}{x^3}\,. \end{cases} \tag{I.5}$$

Für kleine $|x|$ gilt

$$f_j(x) = \frac{2}{3} - \frac{\varepsilon_j}{15} x^2\,; \qquad \varepsilon_1 \equiv \varepsilon_2 \equiv 2\,;\ \varepsilon_3 \equiv 1\,; \qquad |x| \ll 1\,. \tag{I.5'}$$

Ferner ist

$$|f_j(x)| < \tfrac{2}{3}\,; \qquad \text{alle } x \neq 0. \tag{I.5''}$$

Im ungestrichenen Koordinatensystem folgt dann mit (I.3, 4) für die Matrixelemente $n \neq n'$

$$C_{jn_j j' n'}(z) = \sum_{j''} S^{nn'}_{jj''} S^{nn'}_{j'j''} K^{nn'}_{j''}(z), \tag{I.6}$$

mit der Abkürzung

$$K^{nn'}_j(z) = g_0 \sum_{\boldsymbol{k}} \frac{|v_{\boldsymbol{k}}|^2}{|\boldsymbol{k}|^2 - z} \cdot f_j(|\boldsymbol{k}|\, r_{nn'}) \,. \tag{I.7}$$

Zur Auswertung dieses Ausdrucks führen wir die Funktionen ein

$$\begin{cases} \varphi_1(x) = \varphi_2(x) = \dfrac{1}{2}\, e^{ix}\left(-ix + 1 + \dfrac{i}{x}\right) - \dfrac{i}{2x}\,, \\ \varphi_3(x) = e^{ix}\left(-1 - \dfrac{i}{x}\right) + \dfrac{i}{x}\,. \end{cases} \tag{I.8}$$

Es gilt dann

$$f_j(x) = \frac{1}{x^2}\left(\varphi_j(x) + \varphi_j(-x)\right). \tag{I.9}$$

Für kleine $|x|$ ist

$$\varphi_j(x) = -\frac{i}{2\varepsilon_j}\, x\,; \qquad |x| \ll 1. \tag{I.8'}$$

Ferner gilt

$$|\varphi_j(x)/x| < 2\,; \qquad \text{alle } x\,. \tag{I.8''}$$

Nach Übergang zum Kontinuum gemäß (2.30) wird (I.7) unter Berücksichtigung von (2.29, 60)

$$K^{nn'}_j(z) = r_{c0}\, \frac{3}{\pi} \int_0^\infty \mathrm{d}t\, \frac{t^2 u(t)}{t^2 - z}\, f_j(t r_{nn'}) = \frac{r_{c0}}{r^2_{nn'}} \cdot \frac{3}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}t\, \frac{u(t)}{t^2 - z}\, \varphi_j(t r_{nn'})\,. \tag{I.10}$$

(beim letzten Schritt wurde (I.9) und (2.29′) benutzt). Durch Deformation des Integrationsweges in der t-Ebene folgt

$$K^{nn'}_j(z) = 3i\, \frac{r_{c0}}{r^2_{nn'}} \cdot \left(\frac{u(\sqrt{z})}{\sqrt{z}} \cdot \varphi_j(\sqrt{z}\, r_{nn'}) + 2 \sum\nolimits_j \right) \tag{I.11}$$

wo $\sum_j$ die Summe der Residuen von $u(t)\varphi_j(t r_{nn'})/(t^2 - z)$ in den Polen von $u(t)$

in der oberen t-Halbebene ist und man $\sqrt{z}$ so zu wählen hat, daß gilt

(I.12) $$\operatorname{Im}\sqrt{z} > 0\,.$$

Mit der speziellen Abschneidefunktion

(I.13) $$u(t) = \left(1 + \frac{a^2}{4}t^2\right)^{-2},$$

welche offensichtlich (2.25′, 29′) genügt und für (2.38) den Wert

(I.14) $$\beta = 4$$

liefert, wird

(I.15) $$K_0(z) = \frac{r_{c0}}{a}\cdot\frac{(1-(a^2/4)z + ia\sqrt{z})}{(1+(a^2/4)z)^2}\,.$$

Für große Abstände, (2.34), gibt wegen der Beschaffenheit der f_j nur das Intervall $t \lesssim 1/r_{nn'}$ einen merklichen Beitrag zum Integral (I.10). Dort kann man aber $u(t) \cong 1$ setzen. Dann wird (I.11)

(I.16) $$K_j^{nn'}(z) \cong 3i\,\frac{r_{c0}}{r_{nn'}}\cdot\varphi_j(\sqrt{z}\,r_{nn'})/\sqrt{z}\,r_{nn'}\,.$$

Für die Matrix C ist es in diesem Fall zweckmäßig zu schreiben

(I.17) $$C(z) = K_0(z)\cdot 1 + \bar{C}(z)\,,$$

da

(I.17′) $$\bar{C}_{\nu\nu'}(z) \equiv (1-\delta_{nn'})C_{\nu\nu'}(z)$$

für $r_{nn'} \to \infty$ verschwindet.

Im andern Extremfall, (2.35), fällt $u(t)$ ab, bevor sich die f_j merklich ändern. Mit (I.5′) wird

(I.18) $$K_j^{nn'}(z) = K_0(z) - \frac{\varepsilon_j}{10}r_{nn'}^2\left(zK_0(z) + \beta\,\frac{r_{c0}}{a^3}\right).$$

(I.18) eingesetzt in (I.6) gibt mit Hilfe von (I.3, 3′, 5′)

(I.19) $$C(z) \cong K_0(z)\cdot E + \bar{C}\,,$$

wo die Matrix E die Elemente

(I.19′) $$E_{\nu\nu'} = \delta_{jj'}$$

hat und

$$\overline{\overline{C}}_{\nu\nu'}(z) = -\frac{1}{5} r_{nn'}^2 \left(z K_0(z) + \beta \frac{r_{c0}}{a^3} \right) \left(\delta_{jj'} - \frac{x_j^{nn'} x_{j'}^{nn'}}{2 r_{nn'}^2} \right) \tag{I.19''}$$

für $r_{nn'} \to 0$, also auch für $n = n'$, verschwindet.

Zur Auswertung der Determinante (3.10) für die Fälle (2.34, 35) und solche z-Werte, daß gilt

$$C(z)/D_0(z)_{|} \ll 1 ,$$

bzw.

$$\overline{C}(z)/D_0(z)| \ll 1 ; \qquad |\overline{\overline{C}}(z)/K_0(z)| \ll 1 ,$$

benutzen wir die folgende Entwicklung nach dem Parameter η

$$\mathrm{Det}|X + \eta Y| = \mathrm{Det}|X| \cdot \{1 + \eta \cdot \mathrm{Sp}\,(YX^{-1}) + \\ + \tfrac{1}{2}\eta^2 [(\mathrm{Sp}\,YX^{-1})^2 - \mathrm{Sp}\,(YX^{-1})^2] + 0(\eta^3)\} . \tag{I.20}$$

Im Fall (2.34) erhält man in 2. Näherung in η mit

$$X = D_0 \cdot 1 ; \qquad \eta Y = C ; \qquad \mathrm{Sp}\, C = 0$$

und unter Anwendung von (I.3′, 3″)

$$\Delta(z) \cong (D_0(z))^{3Z} \cdot \Big\{ 1 - \sum_{\substack{n \\ (n<n')}} \sum_{n'} \sum_{j} (K_j^{nn'}(z)/D_0(z))^2 \Big\} . \tag{I.21}$$

Dies kann man in derselben Näherung auch schreiben

$$\Delta(z) \cong (D_0(z))^{3Z} \prod_{\substack{n \\ (n<n')}} \prod_{n'} \prod_{j} [1 - (K_j^{nn'}(z)/D_0(z))^2] . \tag{I.21'}$$

(I.21′) ist für $Z = 1$ trivialerweise ($K_j^{nn'} = 0$) exakt richtig, aber auch für $Z = 2$, in einem festen Koordinatensystem, dessen 3. Achse die Richtung von $\boldsymbol{r}_{21}$ hat ($S_{jj'}^{21} = \delta_{jj'}$).

Im Fall (2.35) ist

$$X = 1 + K_0 \cdot E ; \qquad \eta Y = \overline{\overline{C}} ; \qquad X^{-1} = 1 - \frac{K_0}{1 + Z \cdot K_0} \cdot E .$$

Die letzte Beziehung ist leicht zu verifizieren, wenn man beachtet, daß die in (I.19′) definierte Matrix E die Eigenschaft hat

$$E^2 = Z \cdot E . \tag{I.22}$$

(I.20) gibt jetzt in 1. Näherung in η

$$\Delta(z) \cong (1 + ZK_0(z))^3 \cdot \Big\{1 + L(z) \cdot \sum_{\substack{n \\ (n<n')}} \sum_{n'} r_{nn'}^2/a^2\Big\}, \tag{I.23}$$

mit

$$L(z) = K_0(z) \cdot \left(za^2 K_0(z) + \beta \frac{r_{c0}}{a}\right) \Big/ (1 + ZK_0)\,. \tag{I.23'}$$

Anhang II.

Heisenberg-Darstellung.

Da durchwegs die Schrödinger-Darstellung benützt wurde, mag es angezeigt sein, den Zusammenhang zur Heisenberg-Darstellung ([32]) herzustellen. Die Bewegungsgleichungen lauten mit (2.45, 52)

$$\begin{cases} \dfrac{\partial P_k^+(t)}{\partial t} = i\,[H_{\mathrm{Ph}}(P(t), Q(t)), P^+(t)] = -\,(BQ(t))_k\,, \\ \dfrac{\partial Q_k(t)}{\partial t} = i\,[H_{\mathrm{Ph}}(P(t), Q(t)), Q(t)] = P_k^+(t), \end{cases} \tag{II.1}$$

also mit (2.46)

$$\left(\omega_k^2 + \frac{\partial^2}{\partial t^2}\right) Q_k(t) = -\,g_0 \sum_{k'} A_{kk'} Q_{k'}(t)\,. \tag{II.1'}$$

Nun führen wir retardierte und avancierte Green'sche Funktionen ein durch

$$\left(\omega_k^2 + \frac{\partial^2}{\partial t^2}\right) G_k^{\mathrm{av}}{}^{\mathrm{ret}}(t - t') = +\,\delta(t - t')\,, \tag{II.2}$$

$$G_k^{\mathrm{av}}{}^{\mathrm{ret}}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\exp[-i\omega t]}{\omega_k^2 - (\omega \pm i\varepsilon)^2}\,. \tag{II.2'}$$

Diese hängen einerseits mit den entsprechenden invarianten Funktionen zusammen gemäß

$$D_{\mathrm{res}}^{\mathrm{av}}(x) = (2\pi)^{-3} \int \mathrm{d}^3k\, G_k^{\mathrm{av}}{}^{\mathrm{ret}}(t) \exp[i\boldsymbol{kx}]\,,$$

([32]) Vergl. Klein und McCormick ([7]), auch S. S. Schweber: *Nuovo Cimento*, **2**, 397 (1955).

wobei (2.10) zu beachten ist, andrerseits mit den Funktionen (5.32) durch

$$G_k^{\mathrm{av}}{}^{\mathrm{ret}}(t) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\omega_{k'}\{G_{(+)kk'}\exp[\mp i\omega_{k'}t] + G_{(-)kk'}\exp[\pm i\omega_{k'}t]\}\,.$$

Wie bei (5.30) wird (II.2) dazu benützt, um (II.1′) in Form einer Integralgleichung zu schreiben

$$Q_k(t) = Q_k^{(\mp\infty)}(t) - g_0\sum_{k'} A_{kk'}\int_{-\infty}^{+\infty}\mathrm{d}t'\, G_k^{\mathrm{av}}{}^{\mathrm{ret}}(t-t')Q_{k'}(t')\,. \tag{II.3}$$

$Q_k^{(\mp\infty)}(t)$ sind dabei Lösungen der wechselwirkungsfreien Gleichung

$$\left(\omega_k^2 + \frac{\partial^2}{\partial t^2}\right)Q_k^{(\mp\infty)}(t) = 0 \tag{II.4}$$

und somit von der Form

$$Q_k^{(\mp\infty)}(t) = Q_{(+)k}^{\mathrm{aus}}{}^{\mathrm{ein}}\exp[-i\omega_k t] + Q_{(-)k}^{\mathrm{aus}}{}^{\mathrm{ein}}\exp[+i\omega_k t]\,. \tag{II.4′}$$

Im Grenzfall $t\to-\infty$ bzw. $t\to+\infty$ geht $G_k(t)$ über in $G_k^{(-\infty)}(t)$ bzw. $G_k^{(+\infty)}(t)$. Für die Fourierkomponente von (II.3),

$$Q_k(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}t Q_k(t)\exp[+i\omega t]\,, \tag{II.5}$$

welche mit dem transversalen Feld durch

$$\boldsymbol{A}_{\mathrm{tr}}(x) = V^{-\frac{1}{2}}\sum_{\lambda=\pm1}\sum_{\boldsymbol{k}}\boldsymbol{e}_k\int\mathrm{d}\omega Q_k(\omega)\exp[i(\boldsymbol{kx}-\omega t)]$$

zusammenhängt (vergl. (2.39), ferner (2.12) und (IV.6)), folgt mit (II.4′)

$$\begin{cases} Q_k(\omega) = Q_{(+)k}^{\mathrm{aus}}{}^{\mathrm{ein}}\cdot\delta(\omega-\omega_k) - \dfrac{1}{\omega_k^2-(\omega^2\pm i\varepsilon)}\,g_0\sum\limits_{k'}A_{kk'}Q_{k'}(\omega)\,; & \omega>0 \\ Q_k(\omega) = Q_{(-)k}^{\mathrm{aus}}{}^{\mathrm{ein}}\cdot\delta(\omega+\omega_k) - \dfrac{1}{\omega_k^2-(\omega^2\mp i\varepsilon)}\,g_0\sum\limits_{k'}A_{kk'}Q_{k'}(\omega)\,; & \omega<0. \end{cases} \tag{II.6}$$

Die Auflösung dieser Gleichungen geschieht analog wie bei (5.34). Zunächst erhält man mit (2.43) und der Definition (5.37), (5.32)

$$\sum_k \alpha_k Q_k(\omega) = \sum_k (1 + C_{(\pm)k})^{-1}\alpha_k Q_{(+)k}^{\mathrm{aus}}{}^{\mathrm{ein}}\cdot\delta(\omega-\omega_k)\,.$$

Durch Einsetzen in (II.6) wird unter Benützung von (5.38), (5.43)

$$\text{(II.7)} \qquad \begin{cases} Q_k(\omega) = \sum\limits_{k'} M_{(\pm)kk'} Q^{\text{ein}}_{(+)k'} \delta(\omega - \omega_{k'}) ; & \omega > 0 \\ Q_k(\omega) = \sum\limits_{k'} M_{(\mp)kk'} Q^{\text{ein}}_{(-)k'} \delta(\omega + \omega_{k'}) ; & \omega < 0 . \end{cases}$$

Mit (II.5) kann man die Zerlegung von $Q_k(t)$ in positive und negative Frequenzen definieren gemäß

$$\text{(II.8)} \qquad \begin{cases} Q_{(+)k}(t) \equiv \int\limits_0^\infty \mathrm{d}\omega\, Q_k(\omega) \exp[-i\omega t] \\ Q_{(-)k}(t) \equiv \int\limits_{-\infty}^0 \mathrm{d}\omega\, Q_k(\omega) \exp[-i\omega t] \end{cases}$$

und erhält mit (II.7)

$$\text{(II.9)} \qquad \begin{cases} Q_{(+)k}(t) = \sum\limits_{k'} M_{(\pm)kk'} Q^{\text{ein}}_{(+)k'} \exp[-i\omega_{k'} t] \\ Q_{(-)k}(t) = \sum\limits_{k'} M_{(\mp)kk'} Q^{\text{ein}}_{(-)k'} \exp[+i\omega_{k'} t] . \end{cases}$$

Für $t = 0$ folgt die Schrödinger-Darstellung. Insbesondere wird (II.4') und (II.9)

$$\text{(II.10)} \qquad Q^{(\mp\infty)}_k(0) \equiv Q^{\text{aus}}_k = Q^{\text{ein}}_{(+)k} + Q^{\text{ein}}_{(-)k}$$

und

$$\text{(II.11)} \qquad \begin{cases} Q_k(0) \equiv Q'_k = Q'_{(+)k} + Q'_{(-)k} , \\ Q_{(\pm)k}(0) - Q'_{(\mp)k} = \sum\limits_{k'} M_{(\pm)kk'} Q^{\text{ein}}_{(\pm)k'} - \sum\limits_{k'} M_{(\mp)kk'} Q^{\text{aus}}_{(\mp)k} , \end{cases}$$

womit der Zusammenhang mit dem Text, speziell mit (5.11, 12, 24, 46), hergestellt ist.

Man kann den Übergang zwischen den Darstellungen natürlich auch in der Form schreiben

$$\text{(II.12)} \qquad Q_k(t) = \exp[iH_{\text{Ph}}(P', Q')\cdot t] Q'_k \exp[-iH_{\text{Ph}}(P', Q')\cdot t] ,$$

aus welcher (II.1) folgt, oder auch

$$\text{(II.13)} \qquad \begin{cases} Q^{\text{int}}_k(t) \quad \exp[iH_0(P', Q')\cdot t] Q'_k \exp[-iH_0(P', Q')\cdot t] \\ \qquad = U(0, t) Q_k(t)\, U^{-1}(0, t) \end{cases}$$

mit

$$U(0,t) \equiv \exp[iH_0 \cdot t] \cdot \exp[-iH_{\mathrm{Ph}} \cdot t] . \tag{II.13'}$$

Hier ist $Q_k^{\mathrm{int}}(t)$ der Operator in der Wechselwirkungs-Darstellung und genügt den V.R. und der Bewegungsgleichung des freien Feldes, (II.4). Letzteres hat zur Folge, daß zwischen $Q_k^{\mathrm{int}}(t)$ und $Q_k^{(-\infty)}(t)$ eine kanonische Transformation besteht. Durch Spezialisierung auf $t = 0$ erkennt man, daß diese mit (5.25) identisch ist,

$$Q_k^{\mathrm{int}}(t) = A^{-1} Q_k^{(-\infty)}(t) A . \tag{II.14}$$

Analog schließt man durch Vergleich mit (5.46) auf die Beziehung

$$Q_k^{(+\infty)}(t) = S^{-1} Q_k^{(-\infty)}(t) S . \tag{II.15}$$

Indem man die Definition (II.13') auf

$$U(t_0, t_1) = \exp[iH_{\mathrm{Ph}} \cdot t_0] \exp[iH_0 \cdot (t_1 - t_0) \exp[-iH_{\mathrm{Ph}} \cdot t_1] \tag{II.16}$$

verallgemeinert, sodaß gilt

$$U(t_0, t_1) \cdot U(t_1, t_2) = U(t_0, t_2) , \tag{II.16'}$$

erhält man durch Vergleich von (II.13, 14) im Grenzfall $t \to -\infty$

$$A = U(-\infty, 0) = \lim_{T \to \infty} \exp[-iH_{\mathrm{Ph}} \cdot T] \exp[iH_0 \cdot T] \tag{II.17}$$

und weiter

$$Q_k(t) = U^{-1}(-\infty, t) Q_k^{(-\infty)}(t) U(-\infty, t) . \tag{II.18}$$

Endlich ergibt der Vergleich von (II.15, 18) für $t \to +\infty$

$$S = U(-\infty, +\infty) = \lim_{T \to \infty} \exp[-iH_{\mathrm{Ph}} \cdot T] \exp[iH_0 \cdot 2T] \exp[-iH_{\mathrm{Ph}} \cdot T] . \tag{II.19}$$

Die Unitarität von A bzw. S ist eine triviale Folge der rechten Seiten von (II.17) bzw. (II.19).

Anhang III.

Eigenschaften der Matrizen $R_{(\,)}$.

Zunächst folgt aus der Definition (5.37) mit (2.42) und (5.32, 33)

$$C_{(\,)k;\nu\nu'} \equiv g_0 \sum_{k'} \alpha_{\nu k'} \alpha^*_{\nu' k'} G_{(\,)k'k} = C_{(\,)k;\nu'\nu} \equiv (C^T_{(\,)k'})_{\nu\nu'} . \tag{III.1}$$

Wegen

$$\text{(III.2)} \qquad \begin{cases} G^*_{(\pm)kk'} = G_{(\mp)kk'} = -G_{(\pm)k'k} \\ G^*_{(1)kk'} = G_{(1)kk'} = -G_{(1)k'k} \end{cases}$$

und (III.1) gilt ferner

$$\text{(III.3)} \qquad \begin{cases} C_{(\pm)k;\nu\nu'} = C^*_{(\mp)k;\nu'\nu} \equiv (C^+_{(\mp)k})_{\nu\nu'}\,, \\ C_{(1)k,\nu\nu'} = C^*_{(1)k;\nu'\nu} \equiv (C^+_{(1)k})_{\nu\nu'}\,. \end{cases}$$

(die erste dieser Beziehungen erhält man auch direkt aus (3.11)). Aus (5.39) folgt dann mit (2.42) und (III.1)

$$\text{(III.4)} \qquad R^T_{()kk'} \equiv R_{()-k'-k} = g_0\alpha^*_k(1 + C_{()k})^{-1}\alpha_{k'}G_{()k'k}\,.$$

Ferner ist

$$\text{(III.5)} \qquad R^+_{()kk'} \equiv R^*_{()k'k} = g_0\alpha^*_k(1 + C^+_{()k})^{-1}\alpha_{k'}G^*_{()k'k}\,.$$

Aus den letzten beiden Beziehungen folgen aber mit (III.2, 3) die Eigenschaften (5.41, 44).

Bei der Bildung von $R^+_{()}R_{()}$ und $R_{()}R^+_{()}$ treten Produkte der Form $G_{()kk''}G^*_{()k'k''}$ auf, die wir durch Partialbruch-Zerlegung umformen. Wegen

$$\frac{1}{\omega_k^2 - \omega_{k''}^2 \mp i\varepsilon}\cdot\frac{1}{\omega_{k'}^2 - \omega_{k''}^2 \pm i\varepsilon'} =$$

$$= \frac{1}{\omega_k^2 - \omega_{k'}^2 \mp i(\varepsilon + \varepsilon')}\cdot\left(\frac{1}{\omega_{k'}^2 - \omega_{k''}^2 \pm i\varepsilon'} - \frac{1}{\omega_k^2 - \omega_{k''}^2 \mp i\varepsilon}\right)$$

gilt gemäß (5.32) ($\varepsilon > 0$, $\varepsilon' > 0$)

$$\text{(III.6)} \qquad G_{(\pm)kk''}G^*_{(\pm)k'k''} = G_{(\pm)kk'}(G^*_{(\pm)k'k''} - G_{(\pm)kk''})\,.$$

Analog wird, mit der obigen Bedeutung von ε, ε',

$$\text{(III.7)} \qquad \begin{cases} G_{(\pm)kk''}G_{(\pm)k'k''} = G_{(\pm)kk'}\cdot(G_{(\pm)k'k''} - G_{(\pm)kk''})\,; & \varepsilon > \varepsilon' \\ \qquad\qquad = G_{(\mp)kk'}\cdot(G_{(\pm)k'k''} - G_{(\pm)kk''})\,; & \varepsilon < \varepsilon' \end{cases}$$

Daß die rechten Seiten beidemal dasselbe sind, sieht man mit der Beziehung

$$\text{(III.8)} \qquad \overline{G}_{k'k} = \overline{G}_{kk'} \equiv \tfrac{1}{2}(G_{(+)kk'} - G_{(-)kk'}) = i\pi\,\delta(\omega_k^2 - \omega_{k'}^2)\,,$$

welche natürlich im Sinne einer ω_k^2- oder $\omega_{k'}^2$-Integration zu verstehen ist. Mit (5.33), (III.6, 7, 8) folgt dann übereinstimmend aus beiden Formen von (III.7)

$$\text{(III.9)} \qquad G_{(1)kk''}G_{(1)k'k''} = G_{(1)kk'}(G_{(1)k'k''} - G_{(1)kk''}) - \overline{G}_{kk'}\overline{G}_{kk''}\,.$$

(5.39) und (III.5) ergeben jetzt zusammen mit (III.2, 3, 6) und (5.37)

$$\sum_{k''} R^{+}_{(\pm)kk''}R_{(\pm)k''k'} = g_0\alpha_k^*(1 + C^{+}_{(\pm)k})^{-1}(C^{+}_{(\pm)k} - C_{(\pm)k'})\cdot(1 + C_{(\pm)k'})^{-1}\alpha_{k'}G_{(\pm)kk'} .$$

Mit der Identität

$$(1 + C)^{-1}(C - C')(1 + C')^{-1} \equiv -(1 + C)^{-1} + (1 + C')^{-1} \tag{III.10}$$

und mit (III.2) folgt weiter

$$(R^{+}_{(\pm)}R_{(\pm)})_{kk'} = R^{+}_{(\pm)kk'} + R_{(\pm)kk'} , \tag{III.11}$$

d.h. die Beziehung (5.40′).

Ganz analog erhält man mit (III.7) anstelle von (III.6)

$$\sum_{k''} R^{+}_{(-)kk''}R_{(+)k''k'} = g_0\alpha_k^*(1 + C^{+}_{(-)k})^{-1}(C^{+}_{(-)k} - C_{(+)k'})\cdot(1 + C_{(+)k'})^{-1}\alpha_{k'}G_{(\mp)kk'},$$

wobei das obere bzw. untere Vorzeichen von $G_{(\,)kk'}$ durch Anwendung der ersten bzw. zweiten Form von (III.7) folgt. Mit (III.2, 8, 10) erhält man in beiden Fällen übereinstimmend

$$R^{+}_{(-)kk'} + R_{(+)kk'} - (R^{+}_{(-)}R_{(+)})_{kk'} = \\ = 2i\pi g_0\alpha_k^*(1 + C_{(+)k})^{-1}\alpha_{k'}\cdot\delta(\omega_k^2 - \omega_{k'}^2) \equiv 2ir_{(+)kk'} . \tag{III.12}$$

Daraus folgt aber unmittelbar (5.42, 42′).

Auf gleiche Weise wie oben, aber jetzt mit (III.9), wird

$$\sum_{k''} R^{+}_{(1)kk''}R_{(1)k''k'} = g_0\alpha_k^*(1 + C_{(1)k})^{-1}\cdot \\ \cdot\{(C_{(1)k} - C_{(1)k'})G_{(1)kk'} - g_0\sum_{k''}\alpha_{k''}\times\alpha_{k''}^*\bar{G}_{kk''}\bar{G}_{kk'}\}(1 + C_{(1)k'})^{-1}\alpha_{k'} .$$

Daraus folgt mit (III.8, 10)

$$(R^{+}_{(1)}R_{(1)})_{kk'} = R^{+}_{(1)kk'} + R_{(1)kk'} + (r^2_{(1)})_{kk'} , \tag{III.13}$$

wo $r_{(1)}$ analog zur $r_{(+)}$, (III.12), definiert ist, nämlich

$$r_{(1)kk'} \equiv \pi g_0\alpha_k^*(1 + C_{(1)k})^{-1}\alpha_{k'}\delta(\omega_k^2 - \omega_{k'}^2) . \tag{III.13′}$$

Dieses Resultat ist identisch mit (5.45, 45′).

Zum Beweis der Vollständigkeitsrelation (5.40) schreiben wir mit (5.39), (III.5)

$$\sum_{k''} R_{(\pm)kk''}R^{+}_{(\pm)k''k'} = g_0\alpha_k^*\int_0^\infty \mathrm{d}(\omega_l^2)(1 + C_{(\pm)l})^{-1}\cdot \\ \cdot g_0\sum_{k''}\alpha_{k''}\times\alpha_{k''}^*\,\delta(\omega_{k''}^2 - \omega_l^2)(1 + C^{+}_{(\pm)l})^{-1}G_{(\pm)kl}G^{*}_{(\pm)k'l}\cdot\alpha_{k'} .$$

Dies wird unter Berücksichtigung von (5.37) und (III.3, 8, 10)

$$(R_{(\pm)}R^{+}_{(\pm)})_{kk'} = g_0\alpha_k^* \frac{1}{2\pi i}\int_0^\infty \mathrm{d}(\omega_l^2)\cdot\{-(1+C_{(+)l})^{-1}+(1+C_{(-)l})^{-1}\}G_{(\pm)kl}G^*_{(\pm)k'l}\cdot\alpha_{k'}\,,$$

oder mit (III.6, 2)

$$(R_{(\pm)}R^{+}_{(\pm)})_{kk'} = g_0\alpha_k^*\{I_{(\pm)k}-I_{(\mp)k'}\}\alpha_{k'}G_{(\pm)kk'}\,, \tag{III.14}$$

wo die Matrix $I_{(\pm)k}$ definiert ist durch

$$I_{(\pm)k} \equiv \frac{1}{2\pi i}\int_0^\infty \mathrm{d}(\omega_l^2)\{(1+C_{(+)l})^{-1}-(1+C_{(-)l})^{-1}\}G_{(\pm)kl}\,. \tag{III.15}$$

Den letzten Ausdruck formen wir um unter Benützung von (5.32) und einer Definition analog zu (4.6) für die Matrizen $C_{(\pm)l} = C_{(\pm)}(\omega_l^2)$. Nennen wir die Integrationsvariable in (III.15) jetzt x, so wird

$$I_{(\pm)k} = \frac{1}{2\pi i}\int_0^\infty \mathrm{d}x\{(1+C(x+i\varepsilon'))^{-1}-(1+C(x-i\varepsilon'))^{-1}\}\frac{1}{\omega_k^2-(x\pm i\varepsilon'')}\,,$$

wobei der Grenzübergang $\varepsilon'\to 0$ vor $\varepsilon''\to 0$ zu vollziehen ist, sodaß gilt $\varepsilon''\gg\varepsilon'$. Bei Benützung des Integrationsweges von Fig. 4 kann man auch schreiben

$$I_{(\pm)k} = \frac{1}{2\pi i}\int_{\Gamma'}\frac{\mathrm{d}z}{z-(\omega_k^2\mp i\varepsilon)}\cdot(1+C(z))^{-1}\,. \tag{III.15'}$$

Unter der Voraussetzung, daß keines der Matrixelemente $(1+C(z))^{-1}_{\nu\nu'}$ außerhalb der positiven reellen Achse singulär ist, erhält man durch Deformation des Integrationsweges Γ' in Umläufe um $\omega_k^2\pm i\varepsilon$

$$I_{(\pm)k} = -(1+C(\omega_k^2\mp i\varepsilon))^{-1} \equiv -(1+C_{(\mp)k})^{-1}\,. \tag{III.15''}$$

Setzt man dies in (III.14) ein, so folgt mit (5.39), (III.2, 3, 5)

$$(R_{(\pm)}R^{+}_{(\pm)})_{kk'} = R^{+}_{(\pm)kk'} + R_{(\pm)kk'}\,, \tag{III.16}$$

also (5.40). Die oben eingeführte Voraussetzung bedeutet natürlich, daß die Determinante $\Delta(z)$, (3.10), keine Nullstellen außerhalb der positiven reellen Achse haben darf (vergl. Abschnitt 3).

Wir zeigen nun noch, daß die Unitarität der Matrix s, (5.49), unabhängig von der obigen Voraussetzung ist. Wegen (5.42) und (III.12) kann man

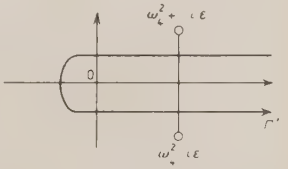

Fig. 4.

auch schreiben

(III.17) $$s = 1 - 2ir_{(+)} .$$

Definiert man analog zu $r_{(+)}$, (III.12), eine Matrix $r_{(-)}$, so gilt

(III.18) $$r_{(-)} = r_{(+)}^{+} .$$

Es folgt

III.19) $$ss^{+} = 1 - r_{(+)} - r_{(-)} + r_{(+)}r_{(-)} .$$

Analog wie oben zeigt man jetzt

(III.20) $$(r_{(+)}r_{(-)})_{kk'} = r_{(+)kk'} + r_{(-)kk'} .$$

Anhang IV.

S-Matrix für klassisches Strahlungsfeld.

Die Bewegungsgleichungen für ein unrelativistisches Punktelektron [33] am Orte $\boldsymbol{q}$ im elektromagnetischen Feld $\boldsymbol{A}$, Φ, lauten in Coulomb-Eichung [34]

(IV.1) $$\ddot{\boldsymbol{A}} - \Delta \boldsymbol{A} = \varrho \dot{\boldsymbol{q}} - \nabla \Phi \equiv \boldsymbol{s}(\boldsymbol{x}, t)$$

(IV.1′) $$\operatorname{div} \boldsymbol{A} = 0$$

(IV.2) $$-\Delta \Phi = \varrho \equiv e\,\delta(\boldsymbol{x} - \boldsymbol{q}(t))$$

(IV.3) $$m\ddot{\boldsymbol{q}} = -e\dot{\boldsymbol{A}}^{\text{ein}}(\boldsymbol{q}) + mr_e\dddot{\boldsymbol{q}} ; \qquad r_e \equiv \frac{2}{3}\frac{e^2}{4\pi m} ,$$

wobei die wesentlichen Züge der klassischen Theorie in der letzten Gleichung stecken. In dieser ist das einlaufende Feld $\boldsymbol{A}^{\text{ein}}(\boldsymbol{x}, t)$ eine Lösung der homogenen Gleichung zu (IV.1) und ergibt zusammen mit der retardierten Lösung $\boldsymbol{A}^{\text{ret}}$

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[33] Wir gehen hier nicht ein auf den Fall einer ausgedehnten Ladung. Vergl. dazu außer Wildermuth [31] z.B. D. Bohm und M. Weinstein: *Phys. Rev.*, **74**, 1789 (1948) (unrelativistisch); E. Bellomo: *Nuovo Cimento*, **2**, 456 (1955) (relativistisch). Bei genügend großer Ausdehnung der Ladung fällt die Schwierigkeit der Selbstbeschleunigung und der Akausalität weg.

[34] Vergl. z. B. W. Heitler: *The Quantum Theory of Radiation*, 3. Auflage (Oxford, 1954), § 1.

von (IV.1) das volle Feld gemäß

$$\boldsymbol{A} = \boldsymbol{A}^{\text{ein}} + \boldsymbol{A}^{\text{ret}} . \tag{IV.4}$$

Der Term $-\nabla\dot{\Phi}$ im Ausdruck für $\boldsymbol{s}$ in (IV.1) kompensiert gerade den Gradient-Anteil des Stromes $\varrho\dot{\boldsymbol{q}}$, sodaß gilt

$$\operatorname{div} \boldsymbol{s} = 0 . \tag{IV.5}$$

Wir gehen wieder in den $\boldsymbol{k}$-Raum und schreiben analog wie in (2.12), wobei aber wegen (IV.1′, 5) nur die transversalen Komponenten $\lambda = \pm 1$ kommen,

$$\boldsymbol{A}(\boldsymbol{x}, t) = V^{-\frac{1}{2}} \sum_{\lambda=\pm 1} \sum_{\boldsymbol{k}} \boldsymbol{e}_{\lambda\boldsymbol{k}} Q_{\lambda\boldsymbol{k}}(t) \exp[i\boldsymbol{k}\boldsymbol{x}] , \tag{IV.6}$$

$$\boldsymbol{s}(\boldsymbol{x}, t) = V^{-\frac{1}{2}} \sum_{\lambda=\pm 1} \sum_{\boldsymbol{k}} \boldsymbol{e}_{\lambda\boldsymbol{k}} s_{\lambda\boldsymbol{k}}(t) \exp[i\boldsymbol{k}\boldsymbol{x}] . \tag{IV.7}$$

Aus (IV.1, 2) folgt dann in der Bezeichnungsweise (2.39, 40)

$$\left(\omega_k^2 + \frac{\partial^2}{\partial t^2}\right) Q_k(t) = s_k(t) \tag{IV.8}$$

(vergl. (II.1′)) und

$$s_k(t) = eV^{-\frac{1}{2}}(\dot{\boldsymbol{q}}\boldsymbol{e}_k) \exp[-i\boldsymbol{k}\boldsymbol{q}(t)] . \tag{IV.9}$$

Diejenige Lösung von (IV.8), welche im $\boldsymbol{x}$-Raum die Gestalt (IV.4) hat, lautet analog zu (II.3)

$$Q_k(t) = Q_k^{(-\infty)}(t) + \int_{-\infty}^{+\infty} \mathrm{d}t' \, G_k^{\text{ret}}(t - t') s_k(t') , \tag{IV.10}$$

wobei das einlaufende Feld wie in (IV.6) definiert ist,

$$\boldsymbol{A}^{\text{ein}}(\boldsymbol{x}, t) = V^{-\frac{1}{2}} \sum_{\boldsymbol{k}} \boldsymbol{e}_k Q_k^{(-\infty)}(t) \exp[i\boldsymbol{k}\boldsymbol{x}] , \tag{IV.6′}$$

und die $Q_k^{(-\infty)}(t)$ der homogenen Gleichung zu (IV.8) genügen.

Wie im quantentheoretischen Fall ist man auch hier auf eine Näherung angewiesen. Wir verlangen, daß das System auf ein zeitlich periodisches einlaufendes Feld ($\sim \exp[-i\omega t]$) ebenso antwortet. Dies ist in der Dipol-Näherung

$$|\boldsymbol{k}\boldsymbol{q}| \ll 1 ; \qquad \exp[\pm i\boldsymbol{k}\boldsymbol{q}] \simeq 1 \tag{IV.11}$$

erfüllt (der mittlere Aufenthaltsort des Elektrons ist dabei im Nullpunkt des Koordinatensystems angenommen). Jetzt kann man in den obigen Gleich-

ungen überall die Zeitabhängigkeit $\sim \exp[-i\omega t]$ setzen. (IV.3) wird aufgelöst durch

$$\boldsymbol{q} = -\frac{i}{e\omega}\cdot\frac{6\pi r_c}{1+i\omega r_c}\cdot V^{-\frac{1}{2}}\sum_k \boldsymbol{e}_k Q_k^{(-\infty)}\,. \tag{IV.3'}$$

(IV.10) geht mit (II.2') über in

$$Q_k = Q_k^{(-\infty)} + \frac{s_k}{\omega_k^2 - (\omega+i\varepsilon)^2}\,, \tag{IV.10'}$$

wobei für das einlaufende Feld gilt

$$(\omega_k^2 - \omega^2)\, Q_k^{(-\infty)} = 0 \tag{IV.12}$$

und

$$s_k = -i\omega e V^{-\frac{1}{2}}(\boldsymbol{q}\boldsymbol{e}_k) \tag{IV.9'}$$

ist. Da nach (IV.12) $Q_k^{(-\infty)} = 0$ ist für $\omega_k \neq \omega$ und wegen $\omega > 0$ kann man mit (5.32) schreiben

$$Q_k = \sum_{k'} M^{\text{klass}}_{(+)kk'} Q_{k'}^{(-\infty)} \tag{IV.13}$$

mit

$$M^{\text{klass}}_{(+)kk'} \equiv \delta_{kk'} - \frac{6\pi r_c}{V}\cdot\frac{(\boldsymbol{e}_k\boldsymbol{e}_{k'})}{1+i\omega_{k'}r_c}\cdot G_{(+)kk'}\,. \tag{IV.13'}$$

Die physikalische Bedeutung von $M^{\text{klass}}_{(+)}$ ist offensichtlich die, daß sich auf eine einfallende linear polarisierte ebene Welle

$$\left\{\begin{aligned} \boldsymbol{A}^{\text{ein}} &= V^{-\frac{1}{2}}\boldsymbol{e}_{k_0}\exp[i(\boldsymbol{k}_0\boldsymbol{x} - \omega t)]\,; \\ Q_k^{(-\infty)} &= \delta_{kk_0}\cdot\exp[-i\omega t] \end{aligned}\right. \qquad |\boldsymbol{k}_0| = \omega \tag{IV.14}$$

(dies erfüllt (IV.12)) ein Feld

$$\left\{\begin{aligned} \boldsymbol{A} &= V^{-\frac{1}{2}}\sum_k \boldsymbol{e}_k M^{\text{klass}}_{(+)kk_0}\exp[i(\boldsymbol{k}\boldsymbol{x} - \omega t)]\,, \\ Q_k &= M^{\text{klass}}_{(+)kk_0}\exp[-i\omega t] \end{aligned}\right. \tag{IV.15}$$

ausbildet.

Zur S-Matrix gelangen wir, indem wir in (IV.13) den Grenzübergang $t \to +\infty$ machen. Zunächst kann man wegen (IV.12) schreiben

$$Q_{k'}^{(-\infty)}(t) = Q_{k'}^{(-\infty)}(0)\exp[-i\omega_{k'}t] = Q_{k'}^{(-\infty)}(0)\exp[-i\omega_k t]\cdot\exp[i(\omega_k - \omega_{k'})t]\,.$$

Nun ist der Ausdruck (vergl. (5.32))

$$(IV.16) \qquad G_{(+)kk'} \exp\left[i(\omega_k - \omega_{k'})t\right] =$$

$$= \frac{i}{\omega_k + \omega_{k'}} \int_0^\infty \mathrm{d}t' \exp\left[-i(t'-t)(\omega_k - \omega_{k'})\right] \exp\left[-t'\varepsilon\right],$$

nach Vertauschen der Grenzübergänge derart, daß zuerst $\varepsilon \to 0$, dann $t \to +\infty$,

$$(IV.16') \qquad = i \frac{\pi}{\omega_k} \delta(\omega_k - \omega_{k'}) .$$

Damit wird (IV.13, 13′) für $t \to +\infty$

$$(IV.17) \qquad Q_k = \sum_{k'} S^{\mathrm{klass}}_{kk'} Q^{(-\infty)}_{k'} ; \qquad t \to +\infty$$

mit

$$(IV.17') \qquad S^{\mathrm{klass}}_{kk'} \equiv \delta_{kk'} - i \frac{\pi}{\omega_k} \cdot \frac{6\pi r_c}{V} \cdot \frac{(\boldsymbol{e}_k \boldsymbol{e}_{k'})}{1 + i\omega_k r_c} \cdot \delta(\omega_k - \omega_{k'}) .$$

Daraus folgt der Wirkungsquerschnitt nach (5.51) mit

$$(IV.18) \qquad \bar{R}^{\mathrm{klass}}_{kk'} = i \frac{\pi}{\omega_k} \cdot \frac{6\pi r_c}{V} \cdot \frac{(\boldsymbol{e}_k \boldsymbol{e}_{k'})}{1 + i\omega_k r_c} ,$$

nämlich

$$(IV.19) \qquad \frac{\mathrm{d}\sigma^{\mathrm{klass}}}{\mathrm{d}\Omega} = \left(\frac{3}{2} r_c\right)^2 \cdot \frac{(\boldsymbol{e}_k \boldsymbol{e}_{k'})^2_{\omega_{k'} = \omega_k}}{1 + \omega_k^2 r_c^2} .$$

Dies ist die Thomson-Formel (vergl. HEITLER [34], § 5).

SUPPLEMENTO AL VOLUME III, SERIE X
DEL NUOVO CIMENTO

N. 3, 1956
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On the Causal Interpretation of Quantum Theories.

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(ricevuto il 16 Febbraio 1956)

CONTENTS. — **1.** Introduction. – **2.** The Bohr-Sommerfeld Theory. – **3.** Schrödinger's Wave Mechanics. – **4.** Heisenberg's Matrix Mechanics. – **5.** The Quantum Theory of Wave Fields.

1. – Introduction.

The discussion concerning the problem as to whether the modern quantum theory may be interpreted in a causal manner has been revived in recent times [1]. It is the purpose of the present work to comment upon this discussion, and to make a small and purely formal contribution towards elucidating this problem by the employment of methods which differ somewhat from those hitherto employed.

A phenomenon in nature is mostly described as « developing causally » if an initial situation A (« cause ») is always followed by the same final situation B (« effect »). This « causality », which dates back to ARISTOTLE, is a prejudice [2] which cannot be done away with, and which hampers the development of natural science and natural philosophy as much as did formerly

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[1] A. EINSTEIN: *Dialectica*, **2**, 320, 312 etc., No. 3, 4 (1948); L. DE BROGLIE: *Phys. Blätt.*, **9**, 541 (1953); E. SCHRÖDINGER: *Brit. Journ. Phil. Sci.*, **3**, No. 10, 11 (1952); M. BORN: *Phys. Blätt.*, **11**, 193 (1955); M. BORN: *Phys. Blätt.*, **11**, 49 (1955); J. L. DESTOUCHES: *Conférences du Palais de la Decouverte*, Sér. A, No. 188 (1953); see also *Journ. Phys.*, **13**, 385 (1952); J. L. DESTOUCHES: *Revue Phil.*, Janvier-Mars 1953 and Presses Univ. France; J. ULLMO: *Rev. Phil.*, 257 (1949); A. EINSTEIN and E. SCHRÖDINGER in: *Festive Publication on the occasion of de de Broglie's 60-th birthday* (Paris, 1952).

[2] A. MARCH: *Die physikalische Erkenntnis und ihre Grenzen* (Braunschweig, 1955), p. 32 ff.

the prejudice of « absolute space ». Even if the relation of uncertainty did not exist and if we had such ideal measuring methods that we would be bound to succeed in repeatedly creating the initial situation A, it would never be possible to prove that such a causality between cause and effect actually exists in the course of the development of phenomena in nature. The final situation B at the time t_1, we must bear in mind, is never and in no case dependent on the cause A (at the time t_0) *alone*: all events E within the space enclosed by the cone of light $t < t_1$, $|x| \leqslant ct$ exercise some influence (even if it be ever so small) upon B (see Fig. 1).

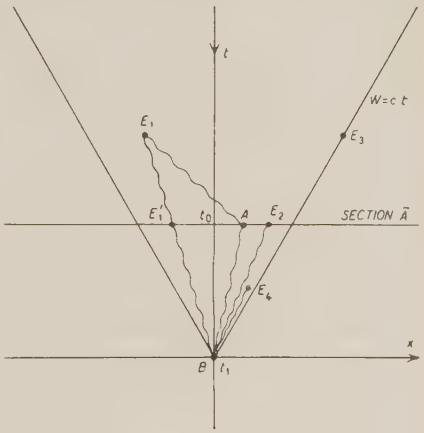

Fig. 1.

Even if we were to succeed in turning back the world's clock, in diminishing entropy, and in arranging all causal chains proceeding from all E and ending in B in the same order in which they had already been arranged in the section $t - t_0$, if, therefore, we were able really to reduce all « causes » brought to bear upon B to the same initial situation $\bar{A}$ that existed at the time t_0, we *might possibly* be in a position of stating that from section $\bar{A}$ there now resulted the same effect B as formerly when the world's clock, on the occasion of the investigation of section $\bar{A}$, also showed the time t_0, and this would *prove nothing whatever*. For, even if we were able to repeat our experiment on so frequent occasions that a conclusion by induction from l to $l + 1$ would be justified, this would be no proof of the existence of a causal behavior of nature defined in the above manner, because a conclusion by induction is no basis for logical argumentation and is never able to furnish proofs and strict laws, but only, at the most, intuitions (3).

We must therefore define the notion « causal (strictly deterministic) development » of a natural development process in a different manner.

All natural processes are, in the last instance, reduceable to the motion of corpuscles and to the propagation of wave fields (*); each of these two

(3) A. March: see (2), p. 28 ff. Compare also the discussion: *Ist die klassische Physik tatsächlich deterministisch?* In Born (1) and the following contribution to discussions in the Physikalische Blätter.

(*) Neither thermodynamics (in any case described by differential equations) nor statistical mechanics and kinetics, atomic physics and physical statistics form an exception.

« basic processes » may be described by means of a differential equation. Thus, it is possible, *in the last instance, to describe all processes in nature by means of differential equations* (*); it will, however, be our task to find out whether such a description is sufficient. The solution of a differential equation is univocally laid down within a certain domain by means of boundary and initial conditions; the functions may also have singularities ([4]).

We will now describe every process that can be described *univocally, completely, and exactly—only* by means of differential equations and boundary— and initial conditions as developing causally. Points of discontinuity ([5]), singularities ([4]), or the occurrence of only discrete values of a quantity ([6]) are of no importance in connection with this definition. In this sense—and this has always been emphasized—Schrödinger's wave mechanics are strictly causal. Bohr gave expression to this in his well-known complementarity according to which the description of atomic phenomena is either (according to HEISENBERG) « illustrative » (anschaulich) but noncausal (uncertainty relation), or (according to SCHRÖDINGER) causal but « non-illustrative » (unanschaulich) (probability interpretation) ([7]). This is also the point of view taken up by SCHRÖDINGER.

Unrelativistic and therefore always *only approximatively correct theories* which do not contain spin, may however, not serve as a basis for the interpretation of processes taking place within the atom any more than the Ptolomaic theory of planetary motion. The theories put forward by SCHRÖDINGER and HEISENBERG are, as we know, fully equivalent to each other, and furnish the same results (which are only approximately valid); they are but serviceable methods of approximation which may, however, not be employed for the purpose of interpreting natural processes. If, however, this is done, it must result

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(*) Neither thermodynamics (in any case described by differential equations) nor statistical mechanics and and kinetics, atomic physics and physical statistics form an exception.

([4]) E.g. Riemann's shock waves in the case of incomplete description because of neglect of inner friction and heat conduct compare: R. SAUER: *Gasdynamik* (Berlin, 1943).

([5]) E.g. resonance curve with nonharmonic oscillators, compare e.g. JOOS: *Lehrbuch der Theoretischen Physik* (Leipzig, 1942), p. 87.

([6]) E.g. eigen oscillations of a membrane, compare e.g. SAUTER: *Differentialgleichungen der Physik, Samml. Göschen* 1070 (Berlin, 1942), p. 111.

([7]) N. BOHR: *Phys. Rev.*, **48**, 696 (1935); *Die Naturwiss.*, **16**, 245 (1928); **17**, 483 (1929); **21**, 13 (1933); *Erkenntnis*, **6**, 293 (1935); W. FURRY: *Phys. Rev.*, **49**, 393, 476 (1936); W. HEISENBERG: *Zeits. f. Phys.*, **43**, 172 (1927).

([8]) E. SCHRÖDINGER: *Brit. J. Phil. Sci.*, **3**, 109, 233 (1952); *Ann. d. Phys.*, **79**, 361, 489, 734 (1926); **80**, 437 (1926); **81**, 109 (1926); L. DE BROGLIE: *Ann. Phys.* **3**, 22 (1925); E. SCHRÖDINGER: *Naturwiss.*, **42**, 720 (1924); further: *The Meaning of Wave Mechanics* (*Louis de Broglie, Physicien et Penseur*), (Paris, 1952); vol. p. le 60e anniversaire de M. de Broglie).

in inner contradictions (the causal theory by Schrödinger and the noncausal theory by Heisenberg are equivalent), and this results in « probability interpretations », « quantum discontinuities », « transition probabilities », etc. ([9]).

We must now enquire after the nature of *genuine* quantum effects, and of the *genuine* deviations from the above defined « causality ».

Let it be assumed that the term « *developing non-causally* » applies to such a process in nature as *cannot* be univocally, completely, and exactly described by differential equations and boundary- and initial conditions. By *quantum effects* we understand the *occurrence of discrete values* with quantities which, in the case of a causal description of the *same* natural process, may assume any value (as e.g. the eigen frequencies of a membrane or the Balmer terms in Schrödinger's theory are *not* quantum effects in this sense, but the Balmer terms in Heisenberg's matrix mechanics would be ascribed as quantum effects in view of the fact that they are produced only by the addition of quantum conditions (commutator relation) to the causal theory).

Soon after the quantum theory was discovered, the idea was conceived that indetermination (« non-causality », statistical causality) simply owes its existence to the fact that there are hidden parameters in nature which cannot be measured by means of the measuring methods available to us today. Therefore we possess but inexact knowledge of the initial state $\bar{A}$, so that we have no reason to be surprised at the inexact statements made concerning the final state. This speculation is known to have been refuted by NEUMANN ([10]); if present-day quantum theories are correct—and this can hardly be doubted in view of the fine successes achieved—such hidden parameters cannot possibly exist, for if this were the case the quantum theories would give different results. Also previous works by DE BROGLIE and by his collaborators on the

([9]) E. SCHRÖDINGER: *Brit. J. Phil. Sci.*, **3**, 109, 233 (1952), see also the same author ([1]); M. BORN: *Zeits. f. Phys.*, **40**, 167 (1926) (statistical interpretation in connection with the many-body problem); compare BORN and FOCK: *Zeits. f. Phys.*, **51**, 165 (1928); A. EINSTEIN: *Phys. Rev.*, **47**, 777 (1933); M. BORN: *Zeits. f. Phys.*, **37**, 863 (1926); **38**, 803 (1926) « Probability Interpretation » compare HEISENBERG: *Zeits. f. Phys.*, **43**, 172 (1927); operator method: P. A. M. DIRAC: *Proc. Roy. Soc.*, A **109**, 642 (1925); A **110**, 561 (1926); A **111**, 281 (1926); A **112**, 674 (1926).

([10]) J. NEUMANN: *Gött. Nachr.* 1929; *Mathematische Grundlagen der Quantenmechanik* (Berlin, 1931); *Commun. Congrès Int. Varsovie* 1938; *Les nouvelles théories de la physique* (Paris, 1939), p. 32; J. SOLOMON: *Journ. Phys.*, (7) **4**, 34 (1933); P. DESTOUCHES-FÉVRIER: *La structure des théories physiques* (Paris, 1951); *Journ. Phys.*, **14**, 640 (1953); see also J. DESTOUCHES: *Journ. Phys.*, **13**, 354 (1952); W. PAULI: Vol. p. le 60^e anniversaire de M. de Broglie (Paris, 1952).

([11]) L. DE BROGLIE: *Compt. Rend.*, **183**, 272, 447 (1926); **184**, 273 (1927); **185**, 380, 1180 (1927) (Older « double solution » works); *Journ. Phys.*, (6) **8**, 225; **7**, 1; W. PAULI: *Reports Solvay Congress* 1928, p. 280 (criticism), see further: L. DE BROGLIE: *Compt. Rend.*, **177**, 517 (1933); **179**, 435 (1924); **180**, 498 (1925); **198**, 135 (1934).

« double solution » have to be looked upon as not quite sound because of Neumann's parameter proof.

However, the parameter proof is valid only for *linear* theories and loses its validity in the case of nonlinear theories. Thus the problem arises as to *whether nonlinear theories are able to furnish a causal interpretation* of quantum effects. As the introduction of nonlinear terms into present-day linear quantum theories corresponds with the assumption of the existence of new and hitherto unknown physical effects (or of another, i.e. of a nonlinear ansatz of already known effects), it could be possible, in this way, to obtain a causal interpretation of quantum effects without being impeded by Neumann's parameter proof.

Apparently on the basis of these deliberations the problem « nonlinearities and quantum effects » has in recent times been repeatedly dealt with ([12]), and also the author occupied himself with this problem by proceeding from a nonlinear meson field theory ([13]). The tests made in this connection will form

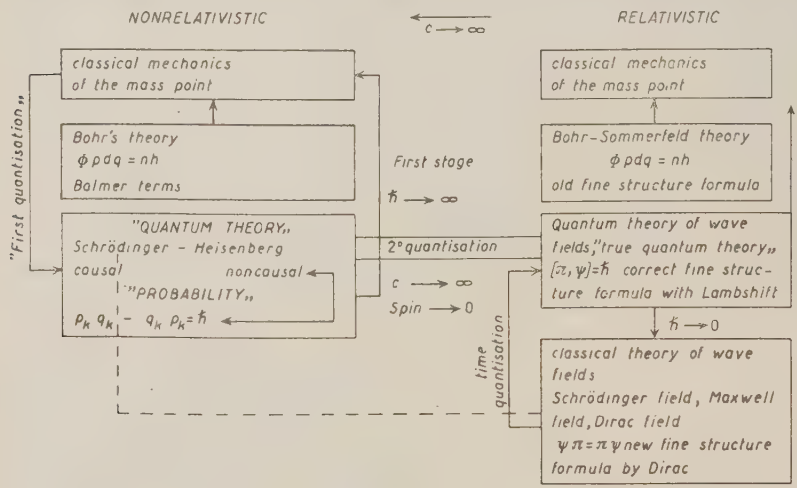

Fig. 2.

([12]) J. MORAN: *Journ. Appl. Phys.*, **22**, 1355 (1951); F. ČAP: *Bornsche Elektrodynamik und nichtlineare pseudoplancksche Oszillatoren* (not published); P. DESTOUCHES and P. FÉVRIER: *Journ. Phys. et Rad.*, **13**, 605 (1952); **13**, 210 (1952); J. DESTOUCHES: *Journ. Phys. et Rad.*, **16**, 81 (1955); **16**, 86 (1955); F. AESCHLIMANN: *Journ. Phys. et Rad.*, **15**, 752 (1954); **15**, 600 (1952); F. ČAP: *Nonlinear Pseudoscalar Meson Theory. EOARDC* Report 633-C (1954); F. ČAP: *Lecture at Munich University*, Inst. f. theoret. Physik (October 1954); *Lecture at Austrian Cultural Center Paris* (October 1955); J. DESTOUCHES: *La Quantification dans la théorie fonctionelle du corpuscule* (1956, in print).

([13]) F. ČAP: *Progr. Theor. Phys.*, **10**, 235 (1953); **13**, 62 (1955); *Phys. Rev.*, **95**, 287 (1954); see also F. ČAP and W. GRÖBNER: *Nuovo Cimento*, **1**, 1211 (1955).

the object of the following report; they by no means claim to be able to give a causal nonlinear interpretation of quantum effects, but merely wish to discuss a possibility which, at first, shall be of a purely formal nature.

Today there exists a number of quantum theories, and we shall examine in each individual case to what extent the introduction of nonlinear terms into the corresponding classical theory (which is produced by the transition $h \to 0$ or matrix $\to$ c-numbers) is able to produce quantum effects (see Fig. 2).

2. – The Bohr-Sommerfeld Theory.

Though today Bohr's theory will hardly ever be used as a basis for the interpretation of atomic phenomena, it may nevertheless be interesting to find out whether it is possible, already in the case of this nearly classical theory, by *omission* of the « grafted » quantum condition $\int p_k \mathrm{d}q_k - n_k h$ and by the introduction of nonlinear hidden parameters, to derive the Balmer formula in a classical-causal manner.

The Balmer formula demands the presence of the electron of the H-atom always only in certain radiationless (*) orbits. Every other non-quantized orbit is prohibited; a stay between the quantum orbits is prohibited ([14]), a change between two orbits takes place within time intervals which are so small that they cannot be noticed. There must therefore exist a hidden damping —or exciting—force by means of which the energy of the system is quickly adapted to the Balmer terms. The computation of this « quantum force K » over the equation of motion of a point-electron in spatial polar coordinates is rather difficult and complicated and will therefore not be dealt with here. We employ a variation method developed by W. Gröbner at the suggestion of the author ([15]).

The quantum force cannot be a conservative force, because it enforces special eigen values E_n, i.e. produces an $E \neq 0$, where $E = T + U = \text{const.}$, T is the kinetic, and U the potential energy. The proof that for conservative forces $\dot{E} = 0$ has already since long been known ([16]).

At the occurrence of non-conservative forces K_k Lagrange's equations of

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(*) This radiationless character of Bohr's orbits can, as we know, not be explained within the framework of Bohr's theory. As we are about to deal only with the derivation of the Balmer formula *without* making use of the quantum condition, we shall leave also this problem open (but compare Bopp's paper in this issue).

([14]) E.g. Sommerfeld: *Atombau und Spektrallinien*, 1. Bd. and compare E. Schrödinger ([1]) and particularly *Naturwiss.*, **12**, 720 (1924).

([15]) W. Gröbner: *Ann. Mat.*, (4) **39**, 11 (1955).

([16]) Compare e.g. Joos ([5]).

motion ([15,16]) are, as we know, as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = K_k \tag{1}$$

and in the case of the hydrogen problem with $\mathcal{L} = T - U$ and

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\dot{\varphi}^2) = \tfrac{1}{2}mv^2\,, \tag{2}$$

$$U = -\frac{e^2}{r}\,, \tag{3}$$

therefore

$$\left\{\begin{aligned} &m\ddot{r} - mr\dot{\vartheta}^2 - mr\sin^2\vartheta\dot{\varphi}^2 - e^2/r^2 = K_r \\ &mr^2\ddot{\vartheta} - mr^2\cos\vartheta\dot{\varphi}^2\sin\vartheta = K_\vartheta \\ &mr^2\sin^2\vartheta\ddot{\varphi} = K_\varphi\,. \end{aligned}\right. \tag{4}$$

If we multiply (4) with $\dot{q}_k$ and add, we find in the well known manner ([15,16])

$$\dot{E} = \sum_k \dot{q}_k K_k\,, \tag{5}$$

(which develops to $\dot{E} = 0$, E — const., for purely conservative forces). It is therefore no longer necessary to integrate (4), but only to determine the K_k in such a manner that from (5) the Balmer terms

$$E_n = -\frac{2\pi^2 me^4}{n^2h^2} \tag{6}$$

result.

This theorem of the conservation of energy which is, as we know, a first integral of the equation of motion (4), assumes the form

$$T + U - R = \text{const} = H = E - R\,, \tag{7}$$

where R is the loss of energy (or the increase of energy), caused by the quantum forces K_k.

Because of (7) it is true that

$$\dot{H} = \dot{T} + \dot{U} - \dot{R} = \dot{E} - \dot{R} = 0\,; \qquad \dot{E} = \dot{R} \tag{8}$$

and thus

$$H = E\,, \quad \text{if} \quad R = 0 \qquad\qquad E_n = \text{const.} \tag{9}$$

In transition orbits (spirals, « jumps ») the electron has the energy H, in stationary (Bohr's) orbits the electron has only the energy E_n:

We now put

$$K_k = \frac{1}{\dot{q}_k} f(E) \tag{10}$$

and with

$$f(E) = \sin \sqrt{\frac{-2\pi^4 m e^4}{h^2 E}}, \tag{11}$$

we obtain from (5) immediately (6). A physical interpretation of the damping force

$$K_k = \frac{1}{\dot{q}_k} \sin \sqrt{\frac{-2\pi^4 m e^4}{h^2((mv^2/2) - e^2/r)}}, \quad v^2 = \sum_k \dot{q}_k^2 \tag{10a}$$

(or of expressions of similar structure as for instance $K_k = \dot{q}_k f(E)$ etc., which also lead to (6)) appears to be but little promising. We are therefore dealing with only a purely formal but nevertheless classical-causal derivation of the Balmer formula. The relativistic theory probably furnishes similar results, but hitherto it has not been investigated.

3. – Schrödinger's Wave Mechanics.

The above theory, according to the point of view adopted by SCHRÖDINGER and DE BROGLIE [8], represents a causal theory. The quantum effects are produced in a purely classical-causal manner by means of eigen values [8]. Schrödinger's theory as a classical field theory [17] demands interpretation of the wave field. No surprise should be caused by the fact that the latter must be described as being physically unreal and as a « probability field », because the theory in question is only approximatively correct: Schrödinger's theory is relativistically not invariant, and it does not contain the electron spin. It therefore by no means does justice to what takes place in nature, and must therefore *not* be employed for the purpose of interpreting atomic phenomena. The relativistic theory (by Dirac) of the electron has a real physical significance which is lost on the occasion of the transition $c \to \infty$.

The fact has often been overlooked that Schrödinger's theory is a causal theory of the probability field and the attempt was made to give a classical interpretation of the quantum effects produced by this « first quantization » (*).

[17] G. FALK and H. MARSCHALL: *Zeits. f. Naturfor.*, **4**a, 131 (1949).

(*) By this mostly the peculiar fact is understood that by $p_k \to (h/i)(\partial/\partial y_k)$ a wave equation for « probability » is obtained. The fact that classical mass points are described by « probability waves » after the carrying out of this process (which supplies discrete energy levels), is called « first quantization ».

These tests, which are based only upon a but approximatively correct, relativistically not invariant, and provisionally interpreted theory, are therefore able to contribute only very little towards the problem of the causal interpretation of quantum effects. Thus, for instance, MADELUNG, FRANKE, BAUER, and others gave a hydrodynamical interpretation of Schrödinger's wave mechanics ([18]), the method of the « double solution » was again adopted in nonlinear form by de Broglie and his collaborators ([19]), while BOHM recently again occupied himself with the problem of hidden parameters, without, however, being able to avoid criticism by PAULI ([20]). BOHM believes to be able to avoid Neumann's parameter proof, but in the end he is not able to do more than to give a formal re-interpretation of Schrödinger's equation. The quantum potential introduced by BOHM produces the quantum effects of the first quantization in the classical manner, and by its own indetermination (hidden parameter!) it causes indeterminism. JÁNOSSY endeavors to interpret quantum effects by velocities which are faster than light ([21]), and WEIZEL gives a classical diffusion model of Schrödinger's theory, by which he hopes to be able to avoid the difficulties caused by hidden parameters ([22]) by the introduction of—in principle—not observable particles (« Zerons »). Other authors rely upon a classical-statistical interpretation, on MARKOV processes ([23]), or, like BOPP, they modify the logics of the classical-statistical motion of particles ([24]) in due appreciation of the difficulty presented by the parameter problem.

([18]) E. MADELUNG: *Zeits. f. Phys.*, **40**, 332 (1926); see also various works published in the *Progr. Theor. Phys.* in the course of recent years; H. FRANKE: *Acta Phys. Hung.*, **4**, 163 (1954); M. BAUER: Various not published works, or works published by the author himself.

([19]) L. DE BROGLIE: *Compt. Rend.*, **233**, 641, 1031 (1951); **234**, 265 (1952); **223**, 874 (1946); **225**, 361 (1947); **229**, 157 (1949); **229**, 269 (1949); see the works by several authors in *L. de Broglie, Physicien et Penseur*, Vol. p. le 60e anniversaire de M. de Broglie (Paris, 1952); see also ([12]), DESTOUCHES; VIGIER: *Compt. Rend.*, **233**, 1031 (1951); **234**, 265 (1952).

([20]) D. BOHM: *Phys. Rev.*, **85**, 166 (1952); **85**, 180 (1952); **89**, 485 (1953); KELLER: *Phys. Rev.*, **89**, 1040 (1953); compare the criticism by PAULI in the de Broglie Festive Publication as cited or K. NAGY: *Acta Phys. Hung.*, **4**, 327 (1954); compare also N. ROSEN and J. ELISHA MITCHEL: *Sci. Soc.*, **61**, No. 1 and 2. August 1945 (precursor of BOHM) and K. NOVOBÁTZKY: *Ann. d. Phys.*, **9**, 406 (1951); compare W. GLASER: *Acta Phys. Austr.*, **4**, 501 (1951).

([21]) L. JÁNOSSY: *Acta Phys. Hung.*, **1**, 423 (1952); and *Ann. d. Phys.*, (6) **11**, 323 (1953).

([22]) W. WEIZEL: *Zeits. f. Phys.*, **134**, 264 (1953) with criticism of BOHM; further: *Zeits. f. Phys.*, **135**, 270 (1953); precursor: J. FENYES: *Zeits. f. Phys.*, **132**, 81 (1952).

([23]) T. TAKABAYASHI: *Progr. Theor. Phys.*, **8**, 143 (1952); **9**, 187 (1953); J. FENYES: *Zeits. f. Phys.*, **132**, 81 (1952); D. BLOHINCEV: *Lehrbuch der Quantenmechanik* (Berlin).

([24]) F. BOPP: *Zeits. f. Naturfor.*, **2**a, 202 (1947); (disproves FENYES); *Zeits. f. Naturfor.*, **9**a, 579 (1954); compare also the discussion: W. MACKE: *Phys. Blätt.*, **11**, 25 (1955).

4. – Heisenberg's Matrix Mechanics.

Though Heisenberg's Matrix Mechanics are fully equivalent with Schrödinger's wave mechanics, the problem is nevertheless interesting, because, like Bohr's theory, this theory is a classical corpuscular theory, upon which a quantum condition (uncertainty relation, commutator relation) is grafted. It may be devoid of interest to know whether by the introduction of quantum forces it is possible, like in the case of Bohr's theory, to derive the (Heisenberg's) quantum effects in the classical manner. Heisenberg's quantum effect is characterized by the fact that by

$$\Delta p_k \Delta q_k \sim h \tag{12}$$

or, which is equivalent ([25]), by

$$p_k q_k - q_k p_k = \hbar \,, \tag{13}$$

discrete eigen values are enforced, and that the quantities p_k, q_k become matrices. In a classical theory ($h \to 0$) the p_k, q_k, are, however, not allowed to be matrices; we must therefore obtain the discrete energy levels *without* the introduction of matrices. As we have already dealt with the hydrogen problem we shall now investigate Planck's harmonic oscillator. It obeys the differential equation

$$m\ddot{q} + kq = 0 \,; \qquad q = A \sin \omega t \,, \tag{14}$$

where q is a matrix, and possesses the energy levels

$$E_n = (n + \tfrac{1}{2})\hbar\sqrt{k/m} \qquad\qquad n = 0, 1, 2, \ldots, \infty. \tag{15}$$

By the introduction of a damping (stabilization ([15])) it is now possible to derive (15) in the classical manner, whereby q, which is now not a matrix, obeys the differential equation

$$m\ddot{q} + kq = F(\dot{q}) \,. \tag{16}$$

Special cases of this sort where frequently investigated. A general discussion

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([25]) See e.g. BECHERT and GERTHSEN: *Atomphysik, Samml. Göschen* (Berlin).

is given by MORAN ([12]); the case

$$F(\dot{q}) = \alpha\dot{q} - \beta\dot{q}^3 , \tag{17}$$

was investigated by HABERLANDNER ([26]); one obtains *one* stable orbit, *one* oval in the phase plane, and one discrete energy value $E = \text{const}\cdot\sqrt{k/m}$.

The case

$$F(\dot{q}) = \alpha\dot{q} - \beta\dot{q}^3 + \gamma\dot{q}^5 , \tag{18}$$

was investigated by KUTTNER ([26]); one obtains *two* stable orbits, *two* ovals in the phase plane, and *two* discrete energy values.

It is recognized that for every zero of $F(\dot{q})$ a discrete eigen value is obtained. As Planck's oscillator possesses an infinite number of zeros according to (15), $F(\dot{q})$ must be a transcendental function.

With

$$F(\dot{q}) = \cos\left(\frac{\pi}{\hbar\sqrt{k/m}}\left[\frac{m}{2}\dot{q}^2 + \frac{k}{2}\left(A^2 - \frac{\dot{q}^2}{k}m\right)\right]\right) \tag{19}$$

$$\left(\text{or } K = \cos\left(\frac{\pi}{\hbar\sqrt{k/m}}E_n\right)\text{according to (10)}\right)$$

one easily obtains (15), as $F(\dot{q})$ vanishes only for

$$\frac{m\dot{q}^2}{2} + \frac{k}{2}\left(A^2 - \frac{\dot{q}^2}{k}m\right) = (n + 1/2)\,\hbar\sqrt{k/m} , \tag{20}$$

which, because of $F(\dot{q}) = 0$ and therefore because of

$$q = A\sin\omega t , \qquad \omega = \sqrt{k/m} \tag{21}$$

develops into

$$\frac{m}{2}\dot{q}^2 + \frac{k}{2}q^2 = E = \frac{kA^2}{2} = (n+1/2)\,\hbar\sqrt{k/m} \tag{22}$$

A physically sensible interpretation of (19) appears to be most doubtful. « Jumps » from one level to another cannot be performed by this theory either in the case of the oscillator, or in the case of the H-atom; the transitions

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([26]) J. HABERLANDNER: *Die stationare Lösung der Differentialgleichung* $yy' = -x - 2(y - y^3/3)$, *home work in mathematics* (Innsbruck University, 1954); P. KUTTNER: *Stationäre Lösungskurven der Differentialgleichung* $yy' = -x - P(y)$, *home work in mathematics* (Innsbruck University, 1955); E. SCHRÖDINGER: *Statistical Thermodynamics* [Cambridge Univ. Press ([9]), ([1])].

$E_2 \to E_1$ take place *very* quickly and must be induced (*). The quantum jumps are unnecessary for the interpretation of atomic phenomena as has been shown by SCHRÖDINGER ([26]).

5. – The Quantum Theory of Wave Fields.

According to an opinion which is widely spread today this theory represents the proper quantum theory. It may be approached in two ways.

4·1. *The Second Quantization.* – In view of the fact that in classical point mechanics there exists no relativistic formulation of the many-body problem, the many-body problem of quantum mechanics was at first approached within the framework of the non-relativistic theories of Heisenberg and Schrödinger. It was necessary, in this connection, to introduce a $3n$-dimensional configuration space in Schrödinger's theory. This transition of n particles to a probability distribution $\psi(x_1 \ldots, x_{3n}, t)$ described by a many-body equation in a causal (but non-illustrative (unanschaulich)) manner was called first quantization. PAULI, JORDAN, KLEIN, and WIGNER then showed ([27]) that by substituting an operator $\psi(x, y, z, t)$ defined in the ordinary space for the function ψ defined in the configuration space, a more simple description is obtained. This substitution $\psi(x_1 \ldots x_{3n}, t) \to \psi(x, y, z, t)$ is mostly called second quantization. In this manner a quantized field theory is obtained.

4·2. *Quantum Theory of Wave Fields.* – In view of the fact that in classical physics wave fields are known (already without first quantization), Dirac deemed it opportune to quantize classical relativistic wave fields $\psi(x, y, z, t)$ ([28]) for the purpose of describing the quanta of these fields (as e.g. of the photon in electrodynamics). As the classical wave field is not quantized, this is the only true quantization. While we are never able to prove the existence of probability fields in nature, we are able to measure Maxwell's fields, electron

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(*) According to a statement made by W. GRÖBNER it is possible by means of suitable ansatzes for $\dot{E} - f(E)$ to obtain also jumps and transition probabilities: if e.g. for $x > x_0$; x_0, the first zero, $f(E) \leqslant 0$ is valid, then $\dot{E}$ is negative right and left beside every zero (with the exception of the first), and the discrete energy levels are labile the slightest disturbance suffices to bring about a « jump » to the next lower level. Transition probabilities could in that case, in accordance with an idea conceived by the author, be expressed by $|f(E)|$.

([27]) W. PAULI and P. JORDAN: *Zeits. f. Phys.*, **47**, 151 (1928); P. JORDAN and K. KLEIN: *Zeits. f. Phys.*, **45**, 751 (1927); P. JORDAN and E. WIGNER: *Zeits. f. Phys.*, **47**, 631 (1928); M. MIE: *Ann. d. Phys.*, (4) **85**, 711 (1928).

([28]) P. A. M. DIRAC: *Proc. Roy. Soc.*, **114** A, 243, 710 (1927).

wave fields, and pion wave fields direct or to prove their existence indirectly (by diffraction, interference, etc.).

If many quanta of the wave field (photons, electrons, mesons) exist, the classical field theory suffices in good approximation for the description of natural phenomena (e.g. the classical derivation of the Coulomb force, of nuclear forces ([29]) etc.); if, however, the corpuscular properties of the field quanta are the *essential* point of issue, the wave field must be quantized.

These wave fields are physically real and can, if electrically neutral, be measured directly; their quantization is the only physically sensible quantization ([30]).

Therefore, Dirac's equation is not the quantumlike description of *one* electron, any more than is the case with Kemmer's or Maxwell's equations, but all these differential equations are *classical field equations* ([31]). Maxwell's equations furnish Coulomb forces and electromagnetic waves, Proca-Kemmer equations furnish meson waves and nuclear forces, Dirac's equation describes the diffraction and interference of electron waves and the resonance oscillations of the electron wave field in a Coulomb field and their frequencies (Dirac's fine structure formula) ([32]).

Only within the framework of this relativistically invariant field theory and its quantum theories,—but never and in no case within the framework of the non-relativistic approximation theories of Bohr, Schrödinger and Heisenberg—is it possible to attempt to understand the genuine quantum effects, the non-causality of atomic phenomena ([33]). All serious causal attempts at interpreting quantum effects or non-causality must therefore be based upon the relativistically invariant quantum theory of wave fields ([31]). The nonrelativistic theories can be derived from the relativistic field theories ([34]); only the latter are also in a position to describe spin phenomena correctly ([35]).

([29]) W. Pauli: *Meson Theory of Nuclear Forces* (New York, 1946); H. Bethe: *Phys. Rev.*, **57**, 260 (1940); F. Čap: *Progr. Theor. Phys.*, **13**, 62 (1955); Bhabha, Sil, Marshak *et al.*: in several works.

([30]) E. Schrödinger, p. 7 ([8]), *The Meaning of Wave Mechanics*; W. Macke: *Phys. Blatt.*, **11**, 25 (1955); E. Schrödinger: *Ann. der Phys.*, (4) **82**, 265 (1927).

([31]) G. Wentzel: *Quantentheorie der Wellenfelder* (Wien, 1943); A. March: *Quantum Theory of Particles and Wave Fields* (New York, 1950); W. Heisenberg and W. Pauli: *Zeits. f. Phys.*, **56**, 1 (1929).

([32]) Compare E. Schrödinger, p. 9 ([9]); E. Schrödinger: *Naturwiss.*, **12**, 720 (1924).

([33]) E. Schrödinger, p. 5, 7 ([30]); p. 24 ([9]); E. Schrödinger: *Ann. der Phys.*, (4) **82**, 265 (1927).

([34]) Compare e.g. the derivation of Schrödinger's equation from the relativistic quantum theory of wave fields: W. Heisenberg and W. Pauli ([31]); see also Gordon and Klein: *Zeits. f. Phys.*, **40**, 117 (1926); **41**, 432 (1927).

([35]) Tetrode: *Zeits. f. Phys.*, **49**, 858 (1928); Möglich: *Zeits. f. Phys.*, **48**, 852 (1928); compare also F. Čap: *Zeits. f. Naturfor.*, **8a**, 740, 747 (1953); *Fortschr. d. Phys.*, **2**, 207 (1955); *Phys. Rev.*, **93**, 907 (1954).

Attempts aiming at the interpretation of genuine quantum effects in the case of special fields ([36]) and at the reduction of a wave field to another wave field have already been undertaken ([37]), but hitherto apparently no attempt has been made to give a causal interpretation of the quantum effects of field theories valid for every spin and for every sort of particle.

For this purpose we shall endeavor to extend the formalism which we employed when dealing with Bohr's theory. As before, we make no attempt at re-interpreting the usual theory, nor is it our intention to modify the theory, and within the permitted quantum states the usual theory shall, as before, remain valid. Only in such states as are prohibited by the quantized wave field theory shall stabilizing forces occur, which in a very short time enforce the adjustment of the quantum state. This stationary state is now characterized by the vanishing of these additional new forces, so that we obtain a theory which is classical, but which, *in the permitted quantum states, is in perfect agreement with the usual quantum theory of wave fields.* All spin indices shall be denoted by σ ([38]); ψ_σ is assumed to be the field function (and/or the quantum-theoretical field operator).

The Lagrangians of the wave fields, like the holonomous scleronomous systems of classical particle mechanics ([15,16]), may be represented by homogeneous-quadratic functions ([31])

$$\mathcal{L} = \sum_{\sigma,\varrho,k,n} a_{kn\varrho\sigma} \frac{\partial \psi_\sigma}{\partial x_k} \frac{\partial \psi_\varrho}{\partial x_n} + \sum_{\sigma,\varrho} b_{\varrho\sigma} \psi_\varrho \psi_\sigma \tag{23}$$

$$\sigma, \varrho = 1 \ldots 2s, \quad s \text{ particle spin}, \quad k, n = 1, 2, 3, 4.$$

(For reasons of simplicity we investigate only real fields; the field quantums are thus electrically neutral).

The Lagrangian equations belonging to (23) are

$$\sum_{k=1}^{4} \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial (\partial \psi_\sigma / \partial x_k)} - \frac{\partial \mathcal{L}}{\partial \psi_\sigma} = K_\sigma \tag{24}$$

and for $\mathcal{L} = T$ and $U = 0$ (« interaction-free wave field ») there results

$$\mathrm{d}\varepsilon = \sum_{\sigma,k} K_\sigma \cdot \frac{\partial \psi_\sigma}{\partial y_k} \mathrm{d}x_k \tag{25}$$

in full analogy to (5).

([36]) H. König: *Acta Phys. Austr.*, **4**, 405 (1951); (interpretations of photons) **5**, 286 (1952); K. Nagy: *Acta Phys. Hung.*, **4**, 327 (1954); J. Fuchs: *Acta Phys. Austr.*, **5**, 349 (1952); *Ann. der Phys.*, **36**, 622 (1939).

([37]) L. Flamm: *Acta Phys. Austr.*, **1**, 259 (1948) and elsewhere W. Glaser: *Zeits. f. Phys.*

([38]) Compare G. Wentzel ([31]), F. Čap ([35]).

For $K_\sigma = 0$ the solution of the field equation (24) is, as we know ([31,35])

$$\psi_\sigma(x,t) = V^{-\frac{1}{2}} \sum_p q_{\sigma p}(t) \exp[ik_p x] = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = V^{-\frac{1}{2}} \sum_p p_{\sigma p}(t) \cdot \exp[-ik_p x] , \tag{26}$$

which, after quantization

$$[\pi, \psi] = \hbar , \tag{27}$$

develops into the operator

$$\psi(x) = V^{-\frac{1}{2}} \sum_p q_{\sigma p} \exp[ik_p x],\ \psi(x,t) = \exp[itH/\hbar]\psi_\sigma(x) \exp -[itH/\hbar] . \tag{28}$$

For the total energy H there results, as we know ([31,35])

$$H = \int_V \mathcal{H} d\tau = \frac{\hbar}{2} \sum_{p,\sigma} \omega_p (2\mathcal{N}_{p\sigma} + 1) , \quad \omega_p = 2\pi\nu_p , \tag{29}$$

where p is the index for various particle momenta—quite analogously to (22).

In order to obtain the genuine quantum effect (29) *without* commutator relation (27), we must chose $K_\sigma \neq 0$ in a suitable manner. By ansatzes analogously to (19) it is possible to obtain (29) by the classical method.

Further examples of such a classical production of quantum effects and attempts aiming at a physical interpretation of the nonlinearities employed are to be discussed in a work which has been planned together with W. Gröbner. The physical interpretation of these new forces is, however, more essential than the purely formal results obtained. As they do not occur in the quantum states, which alone are offered to us by nature, and as they manifest themselves only in the transitions, which take place with the utmost rapidity and, according to the usual theory, in form of jumps, it is extremely difficult and probably quite impossible to prove their existence experimentally. Probably statements concerning the existence or nonexistence of such stabilization and damping forces can be made only after further theoretical deliberations, as e.g. proceeding from the theorem of the conservation of energy and the interaction with other fields. In view of the fact that we are still very much in the dark as regards the theory of radiation damping or the interactions of elementary particles, it is quite possible that certain surprises are still in store for us, possibly also that quantum effects are produced by interaction processes.

For the time being, however, only formal importance can be attached to these investigations. (See, however, the paper by Bopp in this issue).

SUPPLEMENTO AL VOLUME III, SERIE X
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Quantization in the Functional Theory of Particles.

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1. – Physical Systems and Particles.

Actually, the main difficulties of theoretical physics seem to originate in the very notion of a *physical system*. Like an indivisible element, a particle can be geometrically represented by a point (according to the Euclidean definition); but this representation (used both in classical mechanics and in quantum mechanics) may appear to be insufficient to describe a particle from a physical point of view. It is necessary to study how a particle can be described in a theory better than the actual quantum theory.

Theoretical physicists admit that all the parts of the universe make up a whole, and that the motion of any one part reacts upon the whole. But in

practice, certain parts can be separated from the whole, and, for a certain period of time, in some cases to a good approximation, studied independently from the environment. Such parts are called *physical systems*. A part which is indivisible by a given choice of division may be called *a particle*. So, to a first approximation, which is in some cases adequate during a certain period of time, a particle may be considered independent from its environment. In this approximation, it can be considered as a free particle satisfying the laws of wave mechanics.

To a second approximation, the environment is taken into account by means of a given field $F(P, t)$ which is called *external field* (scalar, vector, tensor). Using certain operations, the forces (or the interaction energies) acting on every element of the studied system are derived from this field. This process is in particular applied to the case of a system formed by a single particle, or by n particles.

To a third approximation, the reaction of the whole system upon the field is taken into account, but even this process does not completely describe the real situation, for it neglects effects from the elements of the universe taken individually.

By separating one physical system from the rest of the universe, a fictitious situation is created, which represents the reality in an imperfect way. This separation may be minimized by means of the external field, but not completely suppressed. The quality of approximations may also be improved by considering another larger system of which the primitive system is a part; but even then the universe still remains dissected into separate physical systems.

2. – Functional Representation of a Particle.

In every physical theory, the motion of a system depends only on its own characteristics and on the forces applied to it. In order to avoid the artificial situation of dividing the universe into separate systems, let us use, for a physical system, a process of distinction better than the external field method; in this new process the environment must act not only on the forces applied to the system under consideration, but also on its own characteristics. For this reason, representing a particle by a geometrical point appears insufficient, because such a characterization cannot be altered by the environment. In order to be influenced in its own characteristics, the particle must necessarily be determined by an enumerable infinity of elements [1]. Such a characterization is equivalent to that of a function which belongs to a separable function space (R_a). So we are led to assume that, to a fourth

[1] P. Février: *La structure des théories physiques* (Paris, 1951), p. 21.

approximation, a particle will be represented by a complex valued function u belonging to the space (R_u),

$$u \in (R_u) .$$

But whenever one wishes to specify that the particle has an internal characteristic expressed by an observable with a finite spectrum, (as spin or isotopic spin), then the function u will be decomposed into a finite set of functions u_i. When we consider a set of observers in relative motion to each other and who exchange signals, the variance in the observables attached to the particle can be accurately determined. The number of the components u_i is fixed by the conditions of covariance and depends on the kind of particle under consideration. We shall write u for the set of the functions u_i.

As the geometrical representation holds as a part of the whole representation of physical reality, the particle, since it is an indivisible element, must be geometrically represented by a point M. This point must be definable by means of the function u representing the whole physical character of the particle; then it must be a functional of u, that is

$$M = F[u] .$$

This functional will be denoted by sing, that is

$$M = \text{sing } u .$$

Up to now, the function u has been introduced as a point of a function space (R_u), but not at all as a correspondence between two variables. However, as it is a function, we must designate its argument. We are led to admit [2] that the argument of a function u is a point varying in space-time, that is $u(P, T)$. The function $u(P, T)$ will also be called *physical wave*. To a wave $u(P, T)$ we may associate a fluid giving us an intuitive image of the particle [3].

3. – Calculus of Predictions.

The functional theory of particles must permit the prediction in the form of probabilities concerning an observable B, from the result of an initial measu-

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[2] J. L. Destouches: *Conférence Séminaire de Physique théorique* (Université de Turin, Mars 1952); F. Aeschlimann: *Journ. Phys. Rad.*, **13**. 600 (1952); **15**. 752 (1954); J. L. Destouches: *Funktionnelle Theorie des Elementarteilchens*, Vorlesung Pariser Universitäts-Woche (München, 16 Febr. 1955); P. Février: *L'interprétation physique de la Mécanique ondulatoire et des théories quantiques* (Paris, 1956).

[3] J. L. Destouches: *Journ. Phys. Rad.*, **16**, 81,86 (1955).

rement of an observable A. The general theory of predictions may then be used [4]; it leads us to associate to the result of the initial measurement a set of functionals $X_0[u]$ analogous to the initial wave functions of wave mechanics; and to introduce functionals $X[u\,;t]$ analogous to the wave functions at time t of wave mechanics [5]. A functional $X[u\,;t]$ will be called *functional wave* since it characterizes a wave propagated in the function space (R_u). In order to calculate predictions only for observables depending on position, the geometrical representation of the particle is sufficient, and then, from the functional wave $X[u, t]$, a function $\Psi(M, t)$ can be constructed, where $M = \text{sing}\, u$:

$$\Psi(M, t) = \Psi(\text{ sing } u, t) = \Omega X[u, t]\,,$$

$\Psi(M, t)$ is a point function in space-time; it will be called *predictional wave*. There can be no functional relation between u and Ψ. Hence there appear three types of waves.

In the case of a system of n particles of various types, we have n functions $u_i(P, T)$, n associated fluids, n points $M_i = \text{sing}\, u_i$ and a functional $X[u_1, u_2, ..., u_n; t]$. In the case of particles of the same type, we can equally well consider the fluids of n particles independently or as forming a single fluid with n points M_i associated. Hence we have symmetry and antisymmetry conditions for the functional X.

4. – Realistic Representations.

Since it has been established in general that a phenomenistic predictional theory may always be transformed into a realistic one [6], and conversely [7], realistic images can be used in the functional theory of particle. Then the realistic representation is completely determined by means of the image of the associated fluid, which will satisfy the equations of fluid mechanics. (This fluid corresponds to the physical wave u and has nothing to do with the fluid of probabilities in wave mechanics).

In order to simplify, let us consider the non-relativistic case, without spin. To a function u may be associated a density ϱ and a velocity potential φ as

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[4] J. L. DESTOUCHES: *Corpuscules et systèmes de corpuscules* (Paris, 1941); *Principes fondamentaux de Physique théorique* (Paris, 1942), t. II; P. FÉVRIER: *La structure des théories physiques* (Paris, 1951), pp. 91-143 et pp. 217-258.

[5] P. FÉVRIER: *Journ. Phys. Rad.*, **13**, 605 (1952).

[6] J. L. DESTOUCHES: *Journ. Phys. Rad.*, **13**, 354, 385 (1952).

[7] P. FÉVRIER: *Journ. Phys. Rad.*, **14**, 640 (1953); *L'interprétation physique de la mécanique ondulatoire et des théorie quantiques* (Paris, 1956), pp. 81-92 et 135-150.

follows

$$\varrho = |u|^2, \qquad \varphi = \hbar \arg u ,$$

conversely

$$u = \sqrt{\varrho}\, \exp[ik\varphi] \qquad \text{with } k\cdot\hbar = 1.$$

Provided the pressure depends only on the density, and provided the forces derive from a potential (the case studied by LAGRANGE), the equations of hydrodynamics can be grouped (8) into a single equation in u:

$$(1) \qquad -\frac{\hbar^2}{2m}\Delta u + Vu + i\hbar\frac{\partial u}{\partial t} = u\cdot\left[Q_J(u, \nabla) + \frac{i\hbar}{2|u|^2}\sigma(u, \nabla)\right],$$

where the first member contains the linear terms in u and the second member the non linear terms; as it appears in equations (2) and (3), Q_J is a complementary potential and σ a term of sources; these two terms depend on u and on its derivatives (as it is shown by ∇ as an argument of Q_J and σ). The first member is precisely a Schrödinger's equation for one particle

Let us set

$$u = f\exp[ik\varphi] \qquad \text{with } k\hbar = 1$$

and separate the real part from the imaginary part in (1), we then obtain

$$(2) \qquad \frac{1}{2m}(\operatorname{grad}\varphi)^2 + V - \frac{\hbar^2}{2m}\frac{\Delta f}{f} - \frac{\partial\varphi}{\partial t} = Q_J(f, \varphi, \nabla),$$

$$(3) \qquad \frac{\partial f^2}{\partial t} + \operatorname{div}\left(-\frac{1}{m}f^2\operatorname{grad}\varphi\right) = \sigma(f, \varphi, \nabla).$$

(2) and (3) may be considered as the equations for the fluid associated with the physical wave u: (2) is the equation for the velocity potential of the fluid and (3) is the equation of continuity with sources.

More generally, if

$$E(\psi) = 0$$

denotes a wave equation in wave mechanics (Schrödinger's equation for a system, Klein-Gordon's equation, Dirac's equation, etc.), a better theory will be the following: a functional theory of particles such that, in a realistic representation, the equation satisfied by the function u will be

$$E(u) = Q(u, \nabla),$$

where $Q(u, \nabla)$ is a complex non linear expression of u and its derivatives, whose real part gives a supplementary term of potential, and the complex part a term of sources for the fluid associated with the physical wave u. Conditions of covariance are imposed on Q.

Moreover let us state that the realistic representation corresponding to the fluid associated with the function u, and the representation of a point in a field, according to Mr. Louis de Broglie's double solution theory, must be identical; then, the equation (2) is considered as a Jacobi's equation of a particle in a field f; and (3) is the evolution equation for this field on which the particle reacts; then in this case the real part of Q, i.e. the term Q_J must depend only on the modulus $|u|$ of u, and not on the argument of u ([8]) otherwise the equation (2) would not be a Jacobi's equation; if we impose this supplementary condition, then we shall have

$$Q_J = Q_J(|u|, \nabla). \tag{4}$$

We have not yet been able to determine completely the terms Q_J and σ for till now we know only some of the conditions which they must satisfy ([9]). It is possible that the condition (4) holds experimentally, but on the other hand such a condition might prove inadequate experimentally and it might be necessary to make Q_J depend on u, that is on f and φ, and not only on $f = |u|$; in that case it would be necessary only to give up the interpretation of the particle as a point. But actually this question cannot be decided, and, at least for the moment, the condition (4) can be accepted.

5. – Monochromatic Waves.

The functions u can always be decomposed into modulus and argument

$$u = f \exp[ik\varphi] \qquad (\text{with } k\hbar = 1) \tag{5}$$

(if there are several components, we shall have $u_j = f_j \exp[ik\varphi_j]$).

Among the functions which are solutions of the equation (1) when V does not depend on t and which belong to the space (R_u), we must distinguish those for which the phase φ is linear with respect to time, that is

$$\varphi(M, t) = Et - \varphi_1(M), \tag{6}$$

where E is a constant. Such functions u will be called *monochromatic*, they

([8]) J. L. Destouches: *Journ. Phys. Rad.*, **16**, 84 (1955).

([9]) See Ref.([3]) and *La qualification en théorie fonctionnelle des corpuscules* (Paris, 1956) p. 121.

define *physical monochromatic waves*. In general there are solutions of the form (6) only for certain values of E.

We shall accept as a postulate that, *if a physical system S treated in the approximation of classical mechanics or of wave mechanics is conservative, then, when this system is treated in the approximation of the functional theory, monochromatic functions u exist for certain values of the parameter E.*

(This condition is certainly fulfilled in the double solution theory of Mr. Louis de Broglie, in virtue of his principle of equality for phases). This postulate means that the non linear terms $Q_J(u, \nabla)$ and $\sigma(u, \nabla)$ are such that there exist solutions of the above type.

So the set of functions u is divided into two classes: the class of monochromatic functions and the class of non-monochromatic functions (those for which the phase φ is not of the form (6)).

Concerning a particle acted on by a potential V, in the non-relativistic case without spin, the equation (1) becomes, for a monochromatic wave u_E:

$$\left[-\frac{\hbar^2}{2m}\Delta + V - Q_J(|u_E|, \nabla)\right] u_E - E u_E = \frac{ih}{2|u_E|^2}\, u_E \left[\sigma(u_E, \nabla) - \frac{\partial |u_E|^2}{\partial t}\right]$$

and the real equations (2), (3) become

$$\frac{1}{2m}(\operatorname{grad}\varphi_{1,E})^2 + V - \frac{\hbar^2}{2m}\frac{\Delta f_E}{f_E} - Q_J(f_E, \nabla) - E = 0\ ,$$

$$\frac{1}{2m}(f_E \Delta\varphi_{1,E} + 2\operatorname{grad} f_E \cdot \operatorname{grad}\varphi_{1,E}) + \frac{\partial f_E}{\partial t} = \frac{1}{2f_E}\sigma(f_E, \varphi_{1,E}, \nabla)\ .$$

6. – Waves with Linear Phases.

In the case in which the potential V is null or constant (the particle being not acted on by external forces), we shall suppose besides that *there exist functions u whose phase φ is linear with respect to time and space variables*, that is

$$\varphi = Et - p_x x - p_y y - p_z z + \alpha\ ,$$

where E, p_x, p_y, p_z, α are constants, and $E = (1/2m)(p_x^2 + p_y^2 + p_z^2)$. Such functions u will be considered as defining *waves with linear phases*. (In the double solution theory this condition is fulfilled by virtue of the principle of equality for phases). In the relativistic case $E^2 = m_0^2 c^4 + c^2(p_x^2 + p_y^2 + p_z^2)$.

7. – $\mathcal{C}$-preserving Waves.

Among the waves u we must distinguish the waves having a property which is preserved during the course of time, so that they satisfy the supplementary following condition $\mathcal{C}$:

$$\mathcal{C}\left[u;\ V,\ \mathcal{L}(u);\ x, y, z, t;\ \nabla, \frac{\partial}{\partial t}\right] = 0\ ,$$

where V is the potential and $\mathcal{L}(u)$ denotes the boundary conditions assigned to the function u. An analogous condition would be written concerning a system of particles. A function u satisfying this supplementary condition will be called a $\mathcal{C}$-*preserving* function. An example of such waves is given by the *limit solutions with linear phase* in the case of zero potential, the following condition being assigned to these solutions: the density f^2 of the associated fluid must translate as the point $M(t)$ whose speed is

$$v_M = -\frac{1}{m}\,(\mathrm{grad}\,\varphi)_M\ .$$

So we have the following condition

$$f(\boldsymbol{OP}, t) = f(\boldsymbol{OP} + \boldsymbol{v}_M \Delta t,\ t + \Delta t)\ .$$

Imposing this condition on the u-equations, we find [10] that the source-term is zero:

$$\sigma(u_l, \nabla) = 0\ ,$$

u_l denoting a limit wave with linear phase, a solution of the equation (1). So *a physical wave* u_l, *a limit solution with linear phase, is a* σ-*preserving wave.*

We have proved [11] that, *in the case where* $V = 0$, *the non-linear terms* Q_j *and* σ *may be chosen such that limit solutions with linear phase exist, and such that every linear phase wave tends to a limit solution.* We shall assume as a postulate that, *in the case where* $V = 0$, *the terms* Q_j *and* σ *have been so chosen* (these conditions are necessary in the double solution theory of Mr. Louis de Broglie). The non σ-preserving waves with linear phase are called *transitory waves.*

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[10] J. L. Destouches: *Journ. Phys. Rad.*, **16**, 86 (1955).
[11] J. L. Destouches: *Journ. Phys. Rad.*, **16**, 89 (1955).

8. – Quantization.

Let us call quantified waves, monochromatic waves u which preserve a certain condition $\mathcal{S}$ called *spectral condition*, or *quantization condition*:

$$\mathcal{S}\left[u_E;\, V,\, \mathcal{L}(u_E);\, x, y, z, t;\, \nabla,\, \frac{\partial}{\partial t};\, E\right] = 0\,. \tag{7}$$

The values E for which there exists a monochromatic function u_E satisfying the equation (1), belonging to the space (R_u) and satisfying the spectral condition, will be called *quantized values of energy*. This set of values constitutes the quantized spectrum $\mathcal{A}_E$ of energy.

The spectral condition $\mathcal{S}$ will be supposed such that, to the wave mechanics approximation, the energy spectrum obtained as above is in practice the same as the spectrum given by the quantization rules in wave mechanics; further, that they differ only when usual wave mechanics is no longer adequate, that is, in fact, for large values of energy.

The quantization condition is not uniquely determined, because if $\mathcal{C}(u;\, V,\, \mathcal{L}(u);\, x, y, z, t;\, \nabla,\, \partial/\partial t)$ is a non-zero expression defined for every u of the space (R_u), then $\mathcal{C}\cdot\mathcal{S} = 0$ is equivalent to the condition $\mathcal{S}$. In order to obtain such an expression $\mathcal{C}$, it is sufficient to consider an expression $\mathcal{R}(u;\, V,\, \mathcal{L}(u);\, x, y, z, t;\, \nabla,\, \partial/\partial t)$ which can vanish, and is defined for every u of the space (R_u), and to set $\mathcal{C} = e^{\mathcal{R}}$.

Transformed expressions of $\mathcal{S}$, more general than $\mathcal{C}\cdot\mathcal{S}$, and which admit the same solution, can easily be mentioned: in order to do it, if $\mathcal{S}_1$ is the transform of $\mathcal{S}$ under the transformation Ω, that is

$$\mathcal{S}_1 = \Omega\mathcal{S}\,,$$

the following condition is necessary and sufficient: $\mathcal{S}_1$ vanishes only for the same u_E and the same values of the parameter E as $\mathcal{S}$ does, and their degeneracies are to the same order of multiplicity.

It will be noticed that here the quantization is determined by conditions imposed on the physical waves u characterizing the particles, but not, as in wave mechanics, by conditions imposed on the probability waves.

9. – The case of Zero-Potential.

In the case of a free particle in the whole space without potential, $V = Cte$, we must assume that the spectral condition $\mathcal{S}$ reduces to $\sigma = 0$, that is

$$\mathcal{S}\left(u_{l,p};\, 0,\, \mathcal{L}_\infty(u_{l,p});\, x, y, z, t;\, \nabla,\, \frac{\partial}{\partial t};\, \frac{1}{2m}\,p^2\right) = 0 \longleftrightarrow \sigma(u_{l,p}\,;\, \nabla) = 0\,.$$

The boundary condition $\mathcal{L}_\infty(u)$ expresses that, when the point P, argument of $u(P, t)$, tends to infinity, t being finite, the function u tends to zero in a certain way (for example, P being outside a sphere of sufficiently large radius R, $u(P, t)$ is equal to a square integrable function).

Indeed, to maintain consistency with wave mechanics, it appears that for every value of $\boldsymbol{p}$ we must have linear phase solutions. The limit solutions may be considered as having $\mathcal{C}$-preserving form (for they satisfy the translation condition), and this $\mathcal{C}$-preserving character may be taken as a quantization condition. These limit solutions satisfy $\sigma = 0$; conversely ([12]), from $\sigma = 0$ will be obtained linear phase solutions $u_{l,p}$ for every value of p. So in this case $\sigma = 0$ may be taken as a quantization condition.

10. – The Case of Discontinuous Spectra.

In the particular case above the energy spectrum was continuous; in the general case, the condition of consistency with wave mechanics sets that the spectrum will be mixed, one part being continuous, and the other part discontinuous; besides the spectrum is completely discontinuous in a great number of particular cases, (namely when the particles remain in a bounded domain). Let us examine the case of discontinuous spectrum, whether the whole spectrum is discontinuous or it is a discontinuous part of a mixed one.

Let E_i be a spectral value; there exists at least one monochromatic function u_{E_i} satisfying the spectral condition (7), which corresponds to it. Conversely, if u is a quantified wave function, a value E exists such that the spectral condition (7) is satisfied by u (this value E is equal to that defined by the phase of u).

Till now we do not know an explicit form of the condition $\mathcal{S} = 0$. But the condition of consistency with wave mechanics determines approximately the first spectral values E_i, if these are arranged in increasing sequence.

Let E_i be one of the n known spectral values. There will be at least one function u_i such that

$$\mathcal{S}\left(u_i;\ V,\ \mathcal{L}(u_i);\ x, y, z, t;\ \nabla, \frac{\partial}{\partial t};\ E_i \right) = 0$$

and this equation permits us to determine the monochromatic functions u_i with $E = E_i$ among the solutions of the equation (1). In that case, the spectral condition $\mathcal{S}$ can be written in the form

$$\mathcal{S} = \left\{ \prod_{i\ 1}^{n} \mathcal{S}\left(u;\ V,\ \mathcal{L}(u);\ x, y, z, t;\ \nabla, \frac{\partial}{\partial t};\ E_i \right) \right\} \cdot \mathcal{S}_{1,n}\left(u;\ V;\ \mathcal{L}(u);\ x, y, z, t;\ \nabla, \frac{\partial}{\partial t},\ E \right).$$

([12]) See Ref. ([10]).

$\mathcal{S}_{1,n}$ denoting the expression $\mathcal{S}$ with the following supplementary condition: the values E_i, from $i=1$ to $i=n$, are excluded.

In certain particular cases, the whole discontinuous spectrum may be known, whether we state that the spectrum is the same as the spectrum given by wave mechanics, or we know how it differs from this last spectrum. In that case we should like to write the condition $\mathcal{S}$ in the preceeding form, make n tend to the infinity and pass to the limit; but in general it does not converge. We have seen above that the form of the condition $\mathcal{S}$ is not unique and that $\mathcal{S}\cdot e^{\mathcal{R}}-0$ could also be accepted as a form for $\mathcal{S}$. In these conditions we shall set

$$\mathcal{S}_i=\mathcal{S}\left(u;\ V,\ \mathfrak{L}(u);\ x,y,z,t;\ \nabla,\frac{\partial}{\partial t};\ E_i\right)\exp\left[\mathcal{R}_i\left(u,\ V,\ \mathfrak{L}(u);\ x,y,z,t;\ \nabla,\frac{\partial}{\partial t}\right)\right],$$

$e^{\mathcal{R}_i}$ being a suitable convergence factor. Then the expression $\mathcal{R}$ will be defined as follows:

$$\mathcal{R}\left(u;\ V,\ \mathfrak{L}(u);\ x,y,z,t;\ \nabla,\frac{\partial}{\partial t};\ E_i\right)=\mathcal{R}_i\left(u;\ \ldots;\ \nabla,\frac{\partial}{\partial t}\right),$$

$\mathcal{R}\left(u;\ V,\ \mathfrak{L}(u);\ x,y,z,t;\ \nabla,\frac{\partial}{\partial t};\ E\right)$ — an arbitrary expression finite for $E\neq E_i$ for every i.

and we shall have

$$\mathcal{S}=\prod_{i=1}^{\infty}\mathcal{S}_i=\prod_{i=1}^{\infty}\mathcal{S}(u_E;\ \ldots;\ E_i)\exp\left[\mathcal{R}(u;\ \ldots;\ E_i)\right],$$

for we can set

$$\lim_{n\to\infty}\mathcal{S}_{1,n}=1\,.$$

Indeed, at the limit, $\mathcal{S}_{1,n}$ no longer must contain spectral values, therefore no longer must vanish, and moreover the expressions $\mathcal{R}_i$ have been chosen in order to secure convergence.

11. – Properties of Discontinuous Spectra.

The conditions of connection with usual wave mechanics state first that:

1) *If a spectrum is mixed* (one part discontinuous, one part continuous) *its discontinuous part has at least one point of accumulation.*

2) Conversely, *if a discontinuous enumerably infinite spectrum contains no point of accumulation, then the spectrum has no continuous part.*

But we may have a purely discontinuous spectrum which has points of accumulation (even an infinite number of them). On the other hand, in usual wave mechanics, concerning a purely discontinuous spectrum, we can choose from the set of eigenfunctions an enumerably infinite sequence of functions forming a complete base for the Hilbert space. The condition of consistency gives us the following:

3) *If a spectrum is finite* (that is it contains only n spectral values), *then there is at least one degenerate level of infinite order* (but this case does not occur for energy spectra).

4) *If a discontinuous spectrum does not include any degenerate level for infinite order, then this spectrum is enumerably infinite.*

12. – Indicator Function of a Discontinuous Spectrum.

Let E_i be the i-th level of a discontinuous spectrum without degenerate levels of infinite degree. There exists at least one monochromatic function u_i satisfying the condition

$$\mathcal{S}\left(u_i; V, \mathcal{L}(u_i); x, y, z, t; \nabla, \frac{\partial}{\partial t}; E_i\right) = 0 .$$

To the function u we can associate a variable ζ and to the expression $\mathcal{S}$ an uniform analytic function $S(\zeta)$ such that the zeros of S are the spectral values E_i with the same degree of multiplicity as in the spectrum. This function $S(\zeta)$ will be called an *indicator function of the spectrum.*

If the function $S(\zeta)$ is known, its zeros give the desired spectrum. Conversely if the spectrum is known, we can construct an infinity of indicator functions, each of them having as zeros the spectral values E_i with their own degree of multiplicity, and these values only.

If we know only a part of the spectrum, for example the n first levels, hence we know the first n zeros of the indicator function. We may write:

$$S(\zeta) = S_{P,n}(\zeta) \cdot S_{1,n}(\zeta) ,$$

where $S_{P,n}(\zeta)$ is a function whose only zeros are $E_1, \ldots, E_n$, and $S_{1,n}$ is a function whose zeros are the values E_i with $i > n$.

A function $S_{P,n}(\zeta)$ may be written immediately: if k_i denotes the degree of multiplicity of the spectral value E_i, we shall have, on account of a well known theorem in algebra:

$$S_{P,n}(\zeta) = \prod_{i=1}^{n} (\zeta - E_i)^{k_i} .$$

A function such as $S_{P,n}$ corresponding only to a part of the spectrum will be called *partial indicator*. A partial indicator can, as above, be a polynomial when it corresponds only to a finite number of the levels; on the contrary, it can be a transcendental function if it corresponds to an enumerably infinite part of the spectrum, for example, to the levels E_{2p} of degree $2p$; more generally to levels E_p where $p = f(n)$, n taking on all integral values and $f(p)$ taking on only certain integral values. A partial indicator will also be a function $S(\zeta)$ whose zeros are the spectral values E_i, but not written with their degree of degeneration. Concerning a finite spectrum (then there is at least one degenerate level of infinite degree), a partial indicator of the spectrum will be represented by a polynomial. A function $S_s(\zeta)$ will be called *simple indicator* if its zeros are all the zeros of an indicator $S(\zeta)$, each zero being simple.

13. – The Primary Factors.

Let us consider the case of a discontinuous spectrum without degeneration of infinite degree and without any points of accumulation. Let us assume that the first n spectral values are known. We may use the partial indicator $S_{P,n}$. It is more useful to give it the following equivalent form

$$S^{+}_{P,n}(\zeta) = \prod_{i=1}^{n} \left(1 - \frac{\zeta}{E_i}\right)^{k_i}.$$

We notice that $S_{P,n}(\zeta) \exp[g(\zeta)]$ is yet a partial indicator if $g(\zeta)$ is an entire function (in particular we can pass from $S_{P,n}$ to $S^{+}_{P,n}$ by means of a constant factor e^{λ}). Conversely every entire function which does not vanish for any value of ζ is of the form $\exp[g(\zeta)]$; hence $S^{+}_{P,n}(\zeta) \exp[g(\zeta)]$, where $g(\zeta)$ is an arbitrary entire function, is the general form of the partial indicator $S_{P,n}$. A factor $g_n(\zeta)$ can then be chosen in such a way that, if n tends to infinity, $S^{+}_{P,n}(\zeta) \exp[g(\zeta)]$ tends to an entire function $S(\zeta)$ which is an indicator of the spectrum. There are of course an infinite number of equivalent indicator functions, for, $S(\zeta)$ being one of them, $S(\zeta) \exp[g(\zeta)]$ is another, $g(\zeta)$ being an arbitrary entire function. If we choose a particular indicator function we call it *typical indicator function*.

Then let us consider all the spectral values E_i, arranged according to increasing moduli (in case of positive and negative or complex values in the spectrum). To every non zero value E_i corresponds a polynomial partial indicator

$$\left(1 - \frac{\zeta}{E_i}\right)^{k_i},$$

k_i being the degree of multiplicity of the level E_i.

To this polynomial we associate an entire function S_i partial indicator of the level E_i,

$$S_i = \left(1 \quad \frac{\zeta}{E_i}\right)^{k_i} \cdot \exp\left[g_i(\zeta)\right],$$

where $g_i(\zeta)$ is a polynomial of degree $v(i)$ calculated by the Weierstrass primary factor method ([13]). According to WEIERSTRASS, the function S_i will be called *primary factor.*

The degree $v(i)$ of the polynomial $g_i(\zeta)$ of the convergence factor may be chosen in such a way that the product

$$\prod_{i=1}^{\infty}\left(1-\frac{\zeta}{E_i}\right)^{k_i}\exp\left[g_i(\zeta)\right],$$

is absolutely and uniformly convergent in every circle C with arbitrary radius R and centre the origin.

Then the indicator function $S(\zeta)$ takes the form

$$S(\zeta) = \prod_{i=1}^{\infty} S_i(\zeta) = \prod_{i=1}^{\infty}\left(1-\frac{\zeta}{E_i}\right)^{k_i}\cdot\exp\left[g_i(\zeta)\right].$$

In the case where 0 is a level of degree k_0, we put ζ^k before the product $S(\zeta)$ written above, defined by the infinite product of the primary factors S_i constructed according to the Weierstrass method. $S(\zeta)$ is the indicator of the spectrum under consideration: $S(\zeta)$ vanishes for every spectral value with the appropriate degree of multiplicity, and for these values only.

So, to every discontinuous spectrum without point of accumulation, is associated an indicator which is an entire function.

Now let us assume that, instead of an enumerably infinite spectrum without point of accumulation (i.e. for which, for each integer N, there exists an integer n such that $E_n > N$), we have on the contrary a spectrum with one point of accumulation, for example a spectrum such that

$$\lim_{n\to\infty} E_n = E_\omega .$$

In this case we can again form an uniform analytic function having as zeros only the spectral values E_i, and as an essential singularity the point E_ω of the real axis. In this case, the primary factors method for the values E_i ar-

([13]) WEIERSTRASS: *Ann. École Normale Sup.* t. 8, p. 111 (Paris, 1879); E. GOURSAT: *Cours d'Analyse mathématique* (Paris, VI éd. 1942), t. 11, p. 146; G. VALIRON: *Théorie des fonctions* (Paris, II éd. 1948), p. 427.

ranged in such a way that $|E_i - E_\omega|$ does not increase with i, is again applied but with a slight modification (14). We take as primary factor:

$$S_i(\zeta) = \left(1 - \frac{E_i - E_\omega}{\zeta - E_\omega}\right) \exp\left[h_i(\zeta)\right], \tag{8}$$

where now

$$h_i(\zeta) = g_i\left(\frac{1}{\zeta - E_\omega}\right) = \sum_{k=1}^{\nu-1} \frac{1}{k}\left(\frac{E_i - E_\omega}{\zeta - E_\omega}\right)^k$$

and we obtain an infinite product which converges for every value of ζ different from E_ω.

The function S so obtained is an entire function of $1/(\zeta - E_\omega)$, that is

$$S(\zeta) = G\left(\frac{1}{\zeta - E_\omega}\right),$$

whose unique singularity is the essential singularity E_ω, and whose zeros are the values E_i.

14. – Case of a Finite Number of Points of Accumulation.

If we have n points of accumulation, we can, using transfinite numbers, arrange the spectral values according to increasing moduli, that is

$$E_1,\ E_2,\ \ldots,\ E_m,\ \ldots,\ E_\omega,\ E_{\omega+1},\ E_{\omega+2},\ \ldots,\ E_{2\omega},\ \ldots,\ E_{2\omega+r},\ \ldots,\ E_{n\omega}\ .$$

For every point of accumulation $E_{k\omega}$ we shall have an entire function G_k such that $G_k(1/(\zeta - E_{k\omega}))$ will be a partial indicator (14), and we shall have:

$$S(\zeta) = \prod_{k=1}^{n} G_k\left(\frac{1}{\zeta - E_{k\omega}}\right). \tag{9}$$

As in the case where there is only one point of accumulation, the function $S(\zeta)$ is defined up to a factor $\exp\left[g(\zeta)\right]$, where $g(\zeta)$ is an arbitrary entire function.

But instead of arranging the spectral values E_i according to increasing moduli, it may be preferable to arrange the values in « spectral series », which constitute enumerably infinite sequences, each of them having as a limit one of the points of accumulation. So a spectral value is characterized by two indexes. This modification amounts to taking a finite number of per-

(14) E. PICARD: *Traité d'Analyse* (Paris, III éd. 1926), t. II, p. 143.

mutations of the spectral values in such a way that every E_i of the sequence written above becomes equal to a $E_{j,k}$ such that: 1) the point of accumulation of the sequence of the $E_{j,k}$ is the k-th point of accumulation $E_{k\omega}$; 2) for every j and every p, we have $E_{j,k} < E_{j+p,k}$; 3) for j, k fixed and for every p such that $k+p \leqslant n$, there exists an integer q such that $E_{j,k} < E_{j+q,k+p}$. In this way, $S(\zeta)$ is again the product of n functions as above:

$$S(\zeta) = \prod_{i=1}^{n} G_k^+ \left(\frac{1}{\zeta - E_{\omega,k}} \right),$$

G_k^+ differing from G_k at most by a finite number of primary factors. Each one of these functions G_k^+ corresponds to one of the spectral series. These series will be fixed according to the classification rules of the physicists. A function $S(\zeta)$ written in this form will be called *indicator of the spectrum decomposed into spectral series*. In the case where zero is a spectral value of order k, it is necessary to insert a factor ζ^k.

Hence we see finally that *from every discontinuous spectrum* $\{E_i\}$ (*with or without points of accumulation*) *we can construct an indicator function of the spectrum by the Weierstrass primary factors method and its generalization, whether the spectrum is completely discontinuous, or it is mixed, that is partly discontinuous and partly continuous.*

The use of the indicator function permits a precise classification of discontinuous spectra of energy: 1) completely discontinuous spectra must be distinguished from mixed spectra; 2) concerning their discontinuous parts the spectra must be classified according to the number of their points of accumulation; 3) to every spectrum without point of accumulation and to every spectral series where there is a point of accumulation, an entire function is associated.

The *genus* (according to Laguerre's meaning ([15])) and the *order* of this function characterize the genus and the order of the spectrum. From the genus and the order we obtain indications concerning the way in which the levels $E_{j+1} - E_j$ spread out when j increases. The spectra in the cases which one can calculate in wave mechanics without an approximation method are all of the genus 0 or 1; 4) the spectra can be classified more accurately according to the entire function which constitutes the typical indicator. In this way the spectra of the examples usually given in wave mechanics books are all expressible with the sine function or the Γ-function; we could indicate an example of a spectrum connected to the function $J_{\frac{1}{3}}$ (Bessel's function of order $\frac{1}{3}$).

([15]) E. Borel: *Leçons sur les fonctions entières* (Paris, 1900); R. P. Boas: *Entire functions* (New York, 1954).

15. – Transformation of the Spectral Condition.

The indicators of spectra could be used in any theory with spectra, but they would not play there an essential role because they are not in a direct relation with the formalism employed in these theories (for example in wave mechanics). The situation is not the same in the functional theory of particle, where the indicators of the spectra are closely related to the spectral condition. Let us consider first the case of a spectrum without point of accumulation. The spectral condition, or quantization condition

$$\mathcal{S}\left(u_E;\ V,\ \mathcal{L}(u_E);\ x, y, z, t;\ \nabla, \frac{\partial}{\partial t};\ E\right) = 0\ ,$$

may be transformed as follows if we know one spectral value: let E_i be the i-th spectral value (assumed non zero), and let us set:

$$\Xi\left(u;\ V,\ \mathcal{L}(u);\ x, y, z, t;\ \nabla, \frac{\partial}{\partial t};\ E_i\right) = \tag{10}$$

$$= E_i\left[1 - \mathcal{S}\left(u;\ V,\ \mathcal{L}(u);\ x, y, z, t;\ \nabla, \frac{\partial}{\partial t};\ E_i\right)\right].$$

In the following formula, in order to abbreviate them, we shall no longer write all the arguments, but shall replace them by three dots. According to the transformation above, assuming the value E_i known, the quantization condition becomes

$$\Xi(u;\ \ldots;\ E_i) = E_i\ . \tag{11}$$

In other words, this equation permits us to find the functions u_{E_i} corresponding to the quantized waves for the spectral value E_i. Conversely we have

$$\mathcal{S}(u;\ \ldots;\ E_i) = \left(1 - \frac{\Xi(u;\ \ldots;\ E_i)}{E_i}\right). \tag{12}$$

Given n non zero spectral values numbered E_1, E_2, ..., E_n, we shall have for each of them the relation (10); so for every value E_1, ..., E_n the relation (10) will define Ξ and the equation (11) will give us the quantized functions u.

As it has been noticed above, the quantization condition $\mathcal{S} = 0$ can always be replaced by the product

$$\prod_{i=1}^{n} \mathcal{S}(u;\ \ldots;\ E_i)\cdot\mathcal{S}_{1,n}(u;\ \ldots;\ E_i)\ .$$

Replacing the factors by their expression from (12), we obtain

$$\mathcal{S} = \prod_{i=1}^{n} \left(1 - \frac{\Xi(u; \ldots; E_i)}{E_i}\right) \cdot \mathcal{S}_{1,n}(u; \ldots; E) \,. \tag{13}$$

Let us consider the expression

$$\mathcal{S}^* = \prod_{i=1}^{n} \left(1 - \frac{\Xi(u; \ldots; E)}{E_i}\right) \cdot \mathcal{S}_{1,n}(u; \ldots; E) \,, \tag{14}$$

that differs from $\mathcal{S}$ only in the following way: the parameter E appears as argument of Ξ instead of the value E_i. For $E_j \neq E_1, \ldots, E_n$ there is no difference between (13) and (14). E_i being one of the n values $E_1, \ldots, E_n$, the spectral condition $\mathcal{S} = 0$ reduces to $\Xi(u; \ldots; E_i) = E_i$ (and this happens for each of the n values E_i). Setting $\mathcal{S}^* = 0$ means that one of the factors is zero, then we have

$$\Xi(u; \ldots; E) = E_i \,.$$

In the case where we have suitable solutions u only by setting $E = E_i$, $\mathcal{S} = 0$ and $\mathcal{S}^* = 0$ are equivalent. If, on the contrary, there are solutions $u_{E_i, E_i'}$ for $E = E_i'$ and $E_i \neq E_i$, these solutions must be excluded by stating

$$\Xi(u; \ldots; E) = E_i \quad \text{and} \quad E = E_i \,,$$

i.e. we must demand that the parameter E argument of Ξ takes on no other values than these of the right hand side of the above equation; this supplementary condition being accepted, then $\mathcal{S}^* = 0$ is equivalent to $\mathcal{S} = 0$.

Let us now consider the primary factors of the indicator of the spectrum, constructed according to the Weierstrass' method, and let us modify accordingly the definition of $\mathcal{S}^*$; then we obtain

$$\mathcal{S}^{**} = \prod_{i=1}^{n} \left(1 - \frac{\Xi(u; \ldots; E)}{E_i}\right)^{k_i} \exp\left[g_i(\Xi(u; \ldots; E))\right] \cdot \mathcal{S}_{1,n}(u; \ldots; E) \,. \tag{15}$$

The factors of convergence do not vanish and the condition

$$\mathcal{S}^* = 0 \quad \text{is equivalent to} \quad \mathcal{S}^{**} = 0 \,.$$

The degree of multiplicity of the spectral values does not play any role in the search for quantized functions and we can use the simple indicator (where all the roots are considered as simple).

The spectral condition $\mathcal{S} = 0$ permits us to obtain the spectrum and the

quantized functions u. If n spectral values are already known, the condition $\mathcal{S} = 0$ may be transformed into $\mathcal{S}^{**} = 0$ where $\mathcal{S}^{**}$ is defined by (15), and this can be done for every value of n.

If $\mathcal{S}$ is zero for a function u_E, u_E corresponds to a certain value of E which belongs to the spectrum; let E_i be this value. Then the i-th factor of $\mathcal{S}^{**}$ (taking $n > 1$) will be zero and if we pass to the limit: $G_s(\Xi)$ will be zero, denoting by $G_s(\zeta)$ the simple indicator of the spectrum under consideration. Conversely, if a function u_E satisfies $G_s(\Xi) = 0$, this means that, for a zero E_i of $G_s(\zeta)$, we have $\Xi(u_i; \ldots; E_i) = E_i$; but this is the equation which defines the function u_{E_i}, and hence the function u satisfies $\mathcal{S} = 0$; therefore we obtain the following result:

The spectral condition $\mathcal{S} = 0$ is equivalent to the expression $G_s(\Xi) = 0$, G_s being the simple indicator function of the spectrum:

$$\mathcal{S} = 0 \longleftrightarrow G_s(\Xi) = 0 .$$

But $\mathcal{S}$ has not a uniquely determined form. If $\mathcal{S} = 0$ expresses the spectral condition, it can be replaced by an equivalent condition $\mathcal{S}_1 = 0$, i.e.

$$\mathcal{S} = 0 \longleftrightarrow \mathcal{S}_1 = 0 .$$

Then we can define Ξ by the relation (10) from a certain condition $\mathcal{S}$; so we obtain a new condition $\mathcal{S}_1$ defined by:

$$\mathcal{S}_1(u; \ldots; E) = G_s(\Xi(u; \ldots; E)) .$$

Conversely, if we start from the spectral condition $\mathcal{S}_1 = 0$, whose form is unknown, this condition may always be written in the form

$$G_s(\Xi(u; \ldots; E)) = 0 , \tag{16}$$

where G_s is the simple indicator of the spectrum and Ξ an expression which depends on the same arguments as $\mathcal{S}$, such that $\Xi(u; \ldots; E) - E$, for the quantized u, vanishes only for the zeros of G_s; so $\Xi(u; \ldots; E) = E$ constitutes a new expression of the spectral condition. Hence: *the spectral condition may always be written in the form* (16), *$G_s(\zeta)$ denoting the simple indicator function of the spectrum.*

Instead of the simple indicator function, we may use the indicator function $S(\zeta)$; so we have the same factors, but each of them to the power corresponding to its degree of multiplicity. This introduces a more complicated function, and obliges us to define now in a precise way the meaning of multiplicity of a spectral value.

The preceeding result shows the close connection between the spectral condition and the indicator of the spectrum: this is the fundamental part played by the indicator function in the functional theory of particle. This result implies the following corollary:

If the initial spectral condition $\mathcal{S}_1$ is transformed into an expression $G_1(\Xi)$, where G_1 is an entire function, then, in this case, we obtain the spectrum without computing the quantized functions u: the energy spectrum is given by the zeros of $G_1(\zeta)$.

If we want to calculate the quantized functions u we shall obtain them from an equation for every level E_i, zero of G_1:

$$\Xi(u; \ldots; E_i) = E_i \qquad (E_i \text{ fixed and known}).$$

In this case the spectrum will be obtained directly, but in general the spectral condition initially known cannot be written in the form $G_1(\Xi)$.

16. – The Case of Spectra with Points of Accumulation.

First we must decompose our discontinuous spectrum into spectral series; each of them has a point of accumulation $E_{\omega,k}$, the indices $E_{j,k}$ of the spectral values being arranged as explained above. We can always define an expression Ξ from an expression $\mathcal{S}$ by the formula (10), hence for a spectral value $E_i = E_{j,k}$ we have the condition (11), the relation (12) holding always.

Let us consider the n spectral values $E_{1,k}, \ldots, E_{j,k}, \ldots, E_{n,k}$ of the same spectral series and the corresponding primary factors $\mathcal{S}_i(\zeta)$ defined by (8). The condition (13) is replaced here by

$$\mathcal{S} = \prod_{j=1}^{n} \left(1 - \frac{E_{j,k} - E_{\omega,k}}{\Xi(u; \ldots; E_{j,k}) - E_{\omega,k}}\right) \exp\left[h_{j,k}\left(\Xi_{j,k}(u; \ldots; E_{j,k})\right)\right] \cdot \mathcal{S}_{1,n}(u; \ldots; E)$$

and we shall have, instead of (14), the definition of $\mathcal{S}^*$ obtained from the preceeding relation, replacing $E_{j,k}$ by E in the argument of Ξ. As above, using the same complementary rule we set up the equivalence between $\mathcal{S} = 0$ and $\mathcal{S}^* = 0$.

If a function u satisfies $\mathcal{S} - 0$, it corresponds to a certain spectral series E, and we have two possibilities:

1) it satisfies a primary factor $\mathcal{S}_{j,k} = 0$, and in this case $\mathcal{S} = 0$ has as a consequence

$$S(\Xi) = 0 .$$

Conversely this condition implies that there is an entire function factor

which is zero, and, hence, a primary factor of this function which is zero therefore, and $E_{j,k}$ exists such that

$$\Xi(u_E; \ldots; E) = E \quad \text{and} \quad E = E_{j,k} .$$

2) u_E corresponds to a point of accumulation $E_{\omega,k}$; then the k-th factor function has $E_{\omega,k}$ as an essential singularity, hence S also; in this case we have: $S(\Xi) = S(\alpha)$ where α is an essential singularity of S, and

$$\Xi(u_E, \ldots, E_{\omega,k}) = E_{\omega,k} .$$

Finally, *if a monochromatic function u_E satisfies $\mathcal{S} = 0$, either it implies $S(\Xi) = 0$, or the value E_ω is a point of accumulation of the spectrum.* Conversely, if $S(\Xi) = 0$, then $\mathcal{S} = 0$ and if $\Xi(u, \ldots, E_{\omega,k}) = E_{\omega k}$ for $E = E_{\omega,k}$ as an essential singularity of $S(\zeta)$, then $\mathcal{S} = 0$.

We can also exclude the points of accumulation from the spectrum, i.e. accept as spectrum $\mathscr{A}_E^0$ deduced from $\mathscr{A}_E$ by omitting the points of accumulation. In this case we have a spectral condition $\mathcal{S}^0 = 0$ deduced from $\mathcal{S}$ by excluding the values E, and the functions u_E, for E not belonging to $\mathscr{A}_E^0$ (this condition $\mathcal{S}^0$ is stronger than $\mathcal{S}$ which excludes the values of E which are points of accumulation). So we obtain:

If a function u_E is such that $\mathcal{S}^0 = 0$, then $S(\Xi) = 0$ and conversely $S(\Xi) = 0$ implies $\mathcal{S}^0 = 0$, that is

$$\mathcal{S}^0 = 0 \longleftrightarrow S(\Xi) = 0 \to \mathcal{S} = 0 .$$

17. – The Solvable Case.

Let us assume that the spectral condition

$$\mathcal{S}(u; \ldots; E) = 0 ,$$

is solvable with respect to the parameter E, that is we obtain an expression of the following form:

$$Z\left(u_E; V, \mathcal{L}(u_E); x, y, z, t; \nabla, \frac{\partial}{\partial t}\right) = E , \tag{17}$$

where Z no longer depends on the parameter E (whereas Ξ depended on E). In this case, for every spectral value E, the preceeding condition gives an equation to determine the quantized functions u_E. If we want a general equation holding for every value of E belonging to the spectrum, it will be

deduced from the indicator function by replacing its argument ζ by the expression Z. Indeed, if u_E satisfies (16) for a spectral value E which is not a point of accumulation, one of the primary factors of $S(Z)$ will be zero, therefore we shall have

$$S(Z(u_E; \ldots)) = 0 .$$

Conversely, if this relation holds, Z is equal to a zero of $S(\zeta)$, a primary factor of S is zero, hence the relation (17) is satisfied for a value E of the spectrum.

If E is a point of accumulation, then the expression $S(Z)$ is not defined and Z is an essential singularity of S; hence

$$E \in \mathscr{A}_E^0 \rightarrow [\mathcal{S}(u_E; \ldots; E) = 0 \longleftrightarrow S(Z(u_E; \ldots)) = 0] .$$

E point of accumulation $\rightarrow \mathcal{S}(u_E; \ldots; E) = 0 \longleftrightarrow Z(u_E) = E$.

Thus, *in the solvable case, the general spectral condition for the whole open spectrum* $\mathscr{A}^0$ is given by

$$S(Z(u_E; \ldots)) = 0 , \tag{18}$$

where S *is the indicator of the spectrum.*

Of course the condition $\mathcal{S} = 0$ is not unique therefore Z is not uniquely determined. It will be better to give it the most simple form possible. Since the spectral condition is given by (18), which is a condition of the form $\mathcal{S}_1 = 0$, we can choose for $\mathcal{S}$ a form such that the expression Z does not vanish, except for $E_0 = 0$ belonging to the spectrum. Also $\mathcal{S} = 0$ being a general form of the condition of quantization, we may reduce it to (18) in which Z is a more simple expression. If we can pass from the condition $\mathcal{S} = 0$ to the form (18), then the spectrum is obtained from $S(\zeta) = 0$ without calculating the quantized functions u_E, and these functions can be obtained afterwards by (17) where E is now known. Instead of beginning with the condition $\mathcal{S} = 0$ we may also begin with the equation (11).

18. – Quantization of an Arbitrary Observable.

The considerations developed above for energy may be extended to an arbitrary observable.

Among the physical waves u, we have defined (Sect. **7**) the $\mathcal{C}$-preserving waves, then (Sect. **8**) we have defined the quantized waves as monochromatic waves preserving the spectral condition $\mathcal{S} = 0$.

Let now A be an observable, $\mathscr{A}_A$ its quantized spectrum, and a an element of this spectrum. A wave u is called quantized wave for the value a of A if

it depends on a, that is u_a, and if it satisfies a condition $\mathcal{S}_A = 0$ (depending on a), that is

$$\mathcal{S}_A\left(u_a;\ V, \mathcal{L}(u_a);\ x, y, z, t;\ \nabla, \frac{\partial}{\partial t};\ a\right) = 0\ .$$

The observables such as A for which such a condition $\mathcal{S}_A$ exists, are called *quantized observables*. The observables for which no condition $\mathcal{S}_A$ exists, or for which, if we write such a solution, there exist no solutions u_A belonging to the space (R_u), are called *non-quantized observables*.

The spectrum of a quantized observable A can be continuous or mixed. For these spectra we may develop the same arguments as for the energy spectra; the properties, described in Sect. 11, of discontinuous spectra (whether purely discontinuous spectra, or discontinuous part of a mixed spectrum) hold also for any observable A. For such a spectrum, we can define an *indicator function* $\mathcal{S}_A$ in the same way as for an energy spectrum, and we may classify the spectra according to the properties of the indicator. The spectral condition $\mathcal{S}_A = 0$ can be transformed by means of the indicator, exactly as it had been done for the energy, using an expression Ξ, and in the solvable case, an expression Z. The same cases are to be distinguished and the same formulae to be written.

The preceeding considerations hold not only for an observable of rank 1 (result of measurement a characterized by a single number) but also for an observable of rank n, considering a as a point of the observation space (R_A) of the observable A ([16]); in this case a is an ordered n-uple formed by n numbers which are the coordinates of a in (R_A). If the spectrum of A is discontinuous, the points a of (R_A) can be ordered; moreover each coordinate of a can be considered as the value of an observable of rank 1: let A_i be the observable corresponding to the i-th coordinate of a, then, if the spectrum of A is discontinuous, so is that of A_i, and then we are led to the case above. For A we define an indicator function as a point function, after ordering the points of the spectrum. In particular, in the case of an observable of rank at least 2 which can be decomposed into two observables, the spectrum of A is the direct product of two spectra; in this case we take under consideration the indicators of the spectra of the component observables.

19. – Dependent Observables.

Let A, B be two observables; we say that B *is dependent on* A if we can, for every spectral value a of A, find a value b_a of the spectrum B such that there

([16]) G. Birkhoff and J. von Neumann: *Ann. Math.*, **37**, 823 (1936).

exists at least one function u_{a,b_a} satisfying both spectral conditions $\mathcal{S}_A = 0$ and $\mathcal{S}_B = 0$ of the observables A and B.

Then, for every spectral value a of A, there exists a non void set $\mathcal{E}_{B(a)}$ of spectral values of B compatible with a, and, in the same way, for every spectral value b_a of B, which can be obtained by a function $u_{a,b}$, there exists a set $\mathcal{E}_{A(b_a)}$ of spectral values of A compatible with b_a.

If we know the quantized functions u of the observable A, the calculation of the spectrum of B and of the quantized functions of the observable B is easier, for then we have a part of the quantized functions of B (those which are quantized both for A and for B) and we have already a part of the spectrum of B. Indeed, putting into $\mathcal{S}$ the quantized functions of A, we select the quantized functions both for A and B, and the part of the spectrum of B corresponding to these functions. Having the spectrum of B, we search now for the quantized functions of B for every spectral value.

Conversely, if we know the spectrum of B and the quantized functions u_b of B, by putting the functions u_b into $\mathcal{S}_A$, we select the quantized functions u_{ab} both for A and B; from this we obtain a part of the quantized functions of A (these which are quantized for A and B), and the whole spectrum of A, because for each spectral value b_b which can be obtained from a function u_{ab}, there is one corresponding value of a, and every value a leads to at least one value b.

To sum up, *if the observable B depends on the observable A, giving the quantized functions of A determines a part of the spectrum of B and a part of the quantized functions of B (these quantized for A and B). The datum of the quantized functions of B implies the datum of the whole spectrum of A and of a part of the quantized functions of A (these quantized for A and B); in this case, the calculation of the other quantized functions of A is done the spectrum being already known.*

If the observable B depends on A and A depends on B, the observables A and B are *compatible*. The datum of the quantized functions of one of them implies the datum of the spectrum of the other and a part of its quantized functions.

20. – Observable Function of Another Observable.

A particular case of observables depending one on another is the following: for every value a of the spectrum of A we have

$$b_a = f(a) .$$

In this case, to every spectral value a of A corresponds a single value b_a

(i.e. $\mathcal{E}_{B(a)}$ has only one element b_a). Then, every quantized function u_a for A is also a quantized function for B, and corresponds to the value $b_a = f(a)$ of the spectrum. In this case we say that the observable B is a *function* of the observable A and we write

$$B = f(A) .$$

Now, there is a relation between the indicator functions of the spectra of the two observables. Indeed we have

$$\mathcal{S}_A(u_a; \ldots; a) = 0 \to \mathcal{S}_B(u_a; \ldots; f(a)) = 0 .$$

In particular, in the solvable case, we have

$$Z_A(u_a; \ldots) = a \to Z_B(u_a; \ldots) = f(a) .$$

Hence

$$Z_B(u_a; \ldots) = f(Z_A(u_a; \ldots)) .$$

If $S_A(\zeta)$ is the indicator of the spectrum, then

$$\xi_i = f(\zeta_i) ,$$

is a zero of the indicator S_B of B if ζ_i is the i-th zero of S_A. In the case of a spectrum without point of accumulation for A, and in the case where $|f(|a|)|$ is increasing, $|a|$ being sufficient, we have

$$S_A(\zeta) = \zeta^\alpha \prod_{i=1}^{\infty} \left(1 - \frac{\zeta}{a_i}\right)^{k_i} \exp[g_i(\zeta)]$$

and then

$$S_B(\xi) = g_0(\xi)^\alpha \prod_{i=1}^{\infty} \left(1 - \frac{\zeta}{f(a_i)}\right)^{k_i} \exp[h_i(\xi)] ,$$

if not any $f(a_i)$ is zero. When one $f(a_i)$ is zero, the factor is suppressed in the product and replaced by ξ^{k_i}.

Finally

$$g_0(\xi) \begin{cases} = \xi^\alpha & \text{if } f(0) = 0 \\ = \left(1 - \dfrac{\xi}{f(0)}\right)^\alpha & \text{if } f(0) \neq 0 . \end{cases}$$

One would operate in an analogous way in the case of spectra having points of accumulation.

Then we see that $S_B(f(\zeta))$ is an indicator for A if it is a uniform function. Indeed for every zero ξ_i of S_B we have

$$f(\zeta) = \xi_i \,, \tag{19}$$

for which we obtain $\zeta_{i,1}$, $\zeta_{i,2}$, ..., $\zeta_{i,n}$, values of ζ satisfying the equation above; some of the $\zeta_{i,j}$ belong to the spectrum of A; hence we have factors $1 - (\zeta/\zeta_{i,j})$ that will not be spectral values of A. Then it is possible to construct a suitable analytic uniform function $S_A(\zeta)$ from $S_B(\xi)$ by a process of uniformization when f is a rational function:

$$B = f(A) \to S_A(\zeta) = \boldsymbol{U} S_B(f(\zeta)) \cdot \exp\left[h(\zeta)\right] , \tag{20}$$

where $\boldsymbol{U}$ is an operator to describe the uniformization of $S_B(f(\zeta))$.

If f is an algebraic function, then B is called *observable algebraic function of* A.

Example: Let $B = A^2$, then $+a_i$ and $-a_i$ give the same value $b_i = a_i^2$, S_A includes the factors

$$\left(1 - \frac{\zeta}{a_i}\right)\left(1 + \frac{\zeta}{a_i}\right) \exp\left[g_i^+(\zeta) + g_i^-(\zeta)\right]$$

if $+a_i$ and $-a_i$ belong to the spectrum of A, and we have S_B which includes the factor

$$\left(1 - \frac{\xi}{a_i^2}\right) \exp\left[g_i(\xi)\right] .$$

We see that we pass from S_A to S_B by the formula indicated above: $S_A(\zeta) = S_B(\zeta^2)$; it is sufficient in order to identify the arguments to set $\xi = \zeta^2$, to find again the factors, and to set $g_i(\zeta^2) = g_i^+(\zeta) + g_i^-(\zeta) + h_i(\zeta)$, where $h_1(\zeta)$ is an arbitrary entire function included into $\exp\left[h(\zeta)\right]$.

Let $G(\zeta)$ be a certain entire function. We say that a spectrum of an observable A is *expressible with the entire function* G if we can find a rational function f such that the indicator function S_A of A can be written in the form

$$S_A(\zeta) = G(f_1(\zeta)) \cdot G(f_2(\zeta)) \ldots G(f_p(\zeta)) \,,$$

where $f_1(\zeta)$, $f_2(\zeta)$, ..., $f_p(\zeta)$ are the p solutions of $\zeta - f(x) = 0$.

For example the spectrum of A is expressible by the homographic function $f_h(x)$ with the sine function if we have

$$S_A(\zeta) = \sin f_h(\zeta) \,.$$

The multiplicity of a spectral value of B *with respect to* A is defined as follows:

1) if a value b proceeds from a single value a the multiplicity of B is the same as that of A;

2) if the value b proceeds from n distinct values a_i the multiplicity of the value b is the sum of the multiplicities of every value a_i;

3) if the value b proceeds from an infinity of distinct values a_i, the multiplicity of the value b is infinite.

Let then B be an observable: *the multiplicity of the spectral value b with respect to the set of the observables* is the upper bound of the multiplicity of value b with respect to every observable whose B is a function.

21. – The Usual Notion of Quantization.

According to the theory under consideration, the quantization takes quite different meanings:

1) In the old quantum theory (Bohr-Sommerfeld) the quantization appears as a restriction on the possible motions of the classical mechanics: among them, only the motions satisfying $\int p_i \mathrm{d}q_i - n_i h$ are considered as having a physical meaning: but neither the laws of classical mechanics nor the forces introduced are modified, and the quantization condition amounts to fixing the set of acceptable initial conditions. On the contrary, the electromagnetism is modified by the Bohr's postulate according to which the electron in the quantized orbits does not radiate.

2) In usual wave mechanics, the quantization conditions have a quite different meaning: the quantization is expressed by means of the wave functions (i.e. by means of prediction elements calculated from our information) with a stationary wave condition.

3) In the double solution theory of Louis de Broglie, the quantization is expressed by means of the physical waves u by stationary wave conditions: the function u must be uniform and zero on the boundary of the domain. Then one takes into consideration only its linear approximation:

$$u \sim u_0 + v\,,$$

where u_0 is a singular function of the linear equation which tends toward zero rapidly when moving away from the singularity, while v is a regular solution which alone is taken into account near the boundary of the domain. Then the quantization condition applied to b differs only slightly from the condition applied to the waves ψ of wave mechanics, for v, being a regular solution of the wave equation, is proportional to ψ: we have $\psi = Cv$. Then

the spectra are practically identical in the two theories; only the interpretation varies; here the quantization is expressed by conditions imposed on objective physical elements (functions u), contrary to the case of usual wave mechanics.

22. – The New Notion of Quantization.

We have introduced above a notion of quantization quite different from the usual notion: the particles are physically represented by a function u. Among these functions u, some have a preserving property and satisfy the condition $\mathcal{S}_A = 0$. These functions are said *quantized* for the observable A. In the condition $\mathcal{S}_A$ a parameter a appears which, hence, appears in the wave function, that is $\mathcal{S}_A(u_a; \ldots; a) = 0$. A function u_a is a quantized function for the value a of the observable A.

A condition such as $\mathcal{S}_A = 0$ must be stable, i.e., if we consider a function u' infinitely close to u_a in the sense of the topology of the space (R_u), we must have $\mathcal{S}_A(u'; \ldots; a)$ infinitely small.

This requirement of stability for the spectral condition permits us to consider $\mathcal{S}_A(u_a; \ldots; a) = 0$ as a condition expressing that, for u_a, a certain expression $\mathcal{K}_A(u_a)$ is stationary (i.e. conditions of the first order for the extremum of $\mathcal{K}_A(u_a)$ are satisfied). In particular, for the energy, the condition $\mathcal{S} = 0$ can be considered as expressing the stationary property of an expression $\mathcal{K}$.

We may also consider a spectral condition $\mathcal{S}_A(u_a; \ldots; a) = 0$ as a *conditional first integral*: among the functions u some satisfy a relation depending on one constant only (the spectral value).

The quantization consists then in the following fact, that there are functions u satisfying this conditional integral only for certain values of this constant.

Contrary to the cases of the old quantum theory and of the usual wave mechanics, the non quantized motions are not excluded (i.e. the functions of the space (R_u) which do not satisfy the conditions $\mathcal{S}_A(u_a; \ldots; a) = 0$).

In particular, for the energy, we may have monochromatic functions u_E non quantized. These functions are called *transitory monochromatic functions*. The terms Q_J and σ can be chosen such that these functions tend to the quantized functions.

So a process of emission or absorption of a photon can be described as a process of collision altering abruptly the function u; this function u, at the end of the process, is non quantized and becomes quickly a quantized function according to the transitory process.

Let us consider collisions, i.e. brief interactions between particles. Before the collision we have a certain function u_1 for a particle taken into consideration; this function is in practice quantized if it is monochromatic. After the

collision (that is, when we consider the interaction as ended) we have a function u_2 which is not quantized and which becomes a stationary function. So, processes that usual wave mechanics do not describe (and regard as instantaneous) can be described by the functional theory of particles.

To sum up, the quantized functions u are special functions but they are not the only functions taken into consideration; in other words, the quantized motions are special motions because of a certain stability, but they are not the only possible motions.

23. – The Pointwise Approximation of Geometrical Optics.

The notion of quantization in the functional theory of particles is somewhat difficult. But, to a first approximation, we may restrict ourselves first to a pointwise representation, then to an approximation of geometrical optics. This approximation is obtained from an objectivistic representation with an equation in the form (1). But separating real parts from imaginary parts we obtain a Jacobi's equation (2) and a field equation (3); then we neglect the effect of this field on the particle, i.e. we neglect the quantum potential $-\Delta f/f$ (so we have the approximation of geometrical optics). Jacobi's equation remains valid

$$\frac{1}{2m}(\operatorname{grad}\varphi)^2 + V - Q_J - \frac{\partial\varphi}{\partial t} = 0\,. \tag{21}$$

By neglecting the term Q_J we would obtain the motion in classical mechanics, but, on the contrary, we shall maintain it. The quantized motions are those for which a certain supplementary relation $\mathcal{S}_p = 0$ is satisfied. This relation $\mathcal{S}_p = 0$ is deduced from the condition $\mathcal{S} = 0$ by effecting on $\mathcal{S}$ the same approximation as those used in passing from equation (1) to equation (21) above.

We have the same quantized motions as in the old quantum theory for separable systems if the motions satisfying $\mathcal{S}_a = 0$ are the same as those given by Sommerfeld's quantization conditions.

But we are not compelled to take as quantized motions the motions given by the old quantum theory; on the contrary, we may choose motions such that the spectrum of energy is exactly the same as in usual wave mechanics. Besides, these conditions of quantization are more general than Sommerfeld's conditions, for they hold whether or not the system is separable (taking a system of particles we obtain, with similar approximation, a Jacobi's equation in the configuration space instead of the equation (21), but the results are the same).

Instead of using Jacobi's equation (21), we may also use corresponding

Lagrange's equations, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial q_i'}\right) \quad \frac{\partial T}{\partial q_i} = F_i \quad \text{where} \quad F_i = \frac{\partial (V - Q_J)}{\partial q_i} .$$

Here we have equations which permit a larger application, for they hold even if the forces F_i do not derive from a potential.

In Lagrange's equations the quantization appears in the following way: instead of classical forces only, there is a supplementary term of force. We shall denote by f_i the classical force and by $Q_{c.i}$ the supplementary force. Then, for the quantized motions we shall have $Q_{c.i} = 0$.

This time, instead of expressing the quantization by a restriction on the set of possible motions, as in the old quantum theory of Bohr and Sommerfeld, we take all the motions, but we modify the laws of force in the following way: in the quantized motions, the particles are acted on by classical forces, while supplementary forces act on them when they have non quantized motions, in such a way that these motions are only transitory.

Finally, we have this result:

If we make a pointwise and geometric optical approximation of the functional theory of particles, we obtain the laws of classical mechanics with supplementary forces. To this approximation, the quantization condition consists in making zero the supplementary forces for the quantized motions.

24. – The Theory of Čap.

F. Čap ([17]) has developed a first theory in which he assumes that a particle satisfies the equations of classical mechanics with supplementary force terms which are zero for the quantized motions.

The result given in Sect. 23 can be summarized as follows: *If we make a pointwise and geometric optical approximation of the functional theory of particles, we are led to Čap's pointwise mechanics, and to this approximation the quantization is the same as Čap's quantization condition.*

Nevertheless, on the one hand Čap's original condition is more general, for the supplementary force terms are not considered by him as deriving from a potential, and on the other hand F. Čap makes the particular hypothesis

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([17]) F. Čap: *EOARDEC* (Brussells, 1954). Lecture in Paris, Centre Culturel autrichien, Oct. 1955. Following this lecture, and after discussion with MM. Čap et Gröbner, I thought of defining Indicator Functions of spectra and using them to transform my quantization condition $\mathcal{S} = 0$.

according to which the supplementary force terms $Q_{c,}$ depend only on the velocity. This hypothesis is too narrow and must be slightly generalized, as we see below. However, a theory reduced to point mechanics can be only a first approximation.

25. – The Example of the Oscillator.

Let us consider a harmonic oscillator; with the supplementary forces its equation is:

$$my'' + ky = Q_c .$$

The quantized motions are given by $Q_c = 0$.

Mr. Čap has considered developments into series of powers of Q_c, limiting himself to first terms; hence the first levels obtained are the roots of the approximating polynomial. But, using the indicator function of the spectrum, the term Q_c can be calculated so as to satisfy the required conditions. First, Q_c must be zero for a quantized motion; this requires then that Q_c be a conditional first integral satisfied by a quantized motion. Now, for these motions there is only one uniform integral independent of time, that of energy. This requires that $Q_c = 0$ be a first integral, the same as that of energy, for every quantized motion and for these only. If $S(\zeta)$ is the indicator function of the spectrum (in this example there is no degeneracy), there must be a one-one correspondence between the zeros of $S(\zeta)$ and the quantized motions, i.e. motions such that $Q_c = 0$.

It is sufficient to set

$$Q_c(\mathscr{E}) = S(\mathscr{E})$$

in order to have $\mathscr{E} = E$, for every quantized motion. Let us recall that S is an uniform analytical function and is defined up to a factor $\exp[h(\mathscr{E})]$, $h(\zeta)$ being an entire function. The form of the expression $\mathscr{E}$ must then be determined. The relation above can be satisfied only if $\mathscr{E}$ is a conditional first integral. We are led then to set $\mathscr{E} = T - U$. Besides we can multiply S by an arbitrary factor which is never zero and depends on arbitrary arguments; it is then of the form $e^{\varphi(\mathscr{K})}$ where $\mathscr{K}$ is an arbitrary expression remaining finite for every motion

$$Q_c = S(T - U) \cdot e^{\varphi(\mathscr{K})} .$$

This form of Q_c is general and may be accepted for every system which is conservative from the classical point of view. Here $T - U$ is not a first integral in general (there is not conservation of energy as in the double so-

lution theory of Mr. Louis de Broglie) but it is a conditional first integral: it is a first integral for the quantized motions. The term Q_c must be zero for all these motions and for these only, then it must be expressed by the indicator function of the spectrum, with, as an argument, a conditional first integral, in order to have the relation holding at every moment. This condition requires a modification of Čap's original hypothesis and gives us the general expression of Q_c. On the contrary $\mathcal{K}$ and its argument remain arbitrary and these elements are determined by the condition that the non quantized motions be transitory.

Let us come back to our example; we have

$$T = \tfrac{1}{2}my'^2 \; ; \qquad U = -ky^2 \; ;$$

besides, if we take as quantized levels those given by Planck's rule or Sommerfeld's rule

$$E_i = h\nu \cdot n_i \, , \qquad \text{hence} \qquad S(\zeta) = \frac{1}{\Gamma(-\zeta/h\nu)} \, ,$$

we shall have in this case

$$Q_c = \frac{1}{\Gamma(-(1/h\nu)[\tfrac{1}{2}my'^2 + ky^2])} \, e^{\varphi(\mathcal{K})} \, ,$$

with $h\nu = h\sqrt{k/m}$; the symbol Γ denotes the Eulerian function.

Both the entire function $\mathcal{K}$ and its argument remain arbitrary.

In order to be closer to Čap's hypothesis, we may state that the argument of $\mathcal{K}$ can be chosen in such a way that every non quantized motion tends as fast as is required to a quantized motion. If we expand $e^{\varphi(\mathcal{K})}$ in series, the first terms must give Čap's polynomials.

But we may have as quantized motions those which correspond to the spectra given by wave mechanics. Then we have for $S(\zeta)$

$$S(\zeta) = \frac{1}{\Gamma(\tfrac{1}{2} - \zeta/h\nu)} \, ,$$

hence

$$Q_c = \frac{1}{\Gamma(\tfrac{1}{2} - (1/h\nu)[\tfrac{1}{2}my'^2 + ky^2])} \cdot e^{\varphi(\mathcal{K})} \, .$$

This shows that Čap's method, completed by using the indicators of spectra, gives a better approximation than that given by the old quantum theory.

26. – The Example of the Hydrogen Atom.

Let us treat now the example of a spectrum with one point of accumulation and of a system with several parameters whose variables are separated; the problem of the hydrogen atom is of this type. We obtain the spectrum given by wave mechanics, taking as indicator function the typical indicator; it is suitable to put a factor $C_+(\zeta)$ defined as follows:

$$C_+(\zeta) = \begin{cases} 0 & \text{if} \quad \zeta \geqslant 0 \\ 1 & \text{if} \quad \zeta < 0 \end{cases}$$

in order to incorporate the continuous spectrum obtained for $E > 0$ into the indicator function:

$$S(\zeta) = \sqrt{-\frac{K}{\zeta}} \sin \pi \sqrt{-\frac{K}{\zeta}},$$

we have then

$$Q = C_+(T - U) \sqrt{-\frac{K}{T - U}} \cdot \sin\left(\pi \sqrt{-\frac{K}{T - U}}\right) \cdot e^{\varphi(\mathcal{H})},$$

$T - U$ denoting here the energy of the relative motion around the center of gravity, m being the reduced mass of electron, and

$$K = \frac{2\pi^2 m e^4}{\hbar^2}.$$

The function $\mathcal{H}$ is chosen such that the non quantized motions are transitory motions tending arbitrarily fast to a quantized motion. We can take r', derivative of the vector radius, as argument of $\mathcal{H}$. The motions with positive total energy corresponding to hyperbolic paths are quantized motions (in the continuous spectra meaning) and Q_c is zero here because of the factor $C_+(T - U)$. These motions are identical to classical motions.

27. – The Theory of Bopp.

Some phenomena can be treated as purely mechanical, but a great number require electromagnetic considerations; Bopp has proposed linear modifications of the electromagnetism equations such that, in a quantized orbit, the electron does not radiate, while non quantized motions exist in which the

electron radiates and tends rapidly to a quantized motion. The quantization condition is in the form $\mathcal{S} = 0$ as in the functional theory, but, as the approximation is pointwise, the arguments of the condition are the vector coordinate q of the electron and the derivative operator $\mathrm{d}/\mathrm{d}t$. In Bopp's theory, $\mathcal{S}$ is a linear expression formed by the derivative of odd order of q from the 3rd order, that is

$$\mathcal{S}_B\left(\dddot{q}, \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) = 0 .$$

28. – Quantization and Wave Mechanics.

Naturally the purely pointwise theories can be only, either partial theories (concerning only a part of the phenomena), or approximate theories.

Let us come back to the functional theory of particles. We may obtain another approximation of this theory by neglecting the transitory phenomena and taking only into consideration the quantized motions.

Suppose we have a certain system S. The quantization condition $\mathcal{S}_A = 0$ gives for the observable A a certain spectrum which can be represented by its indicator function $\mathcal{N}_A(\zeta)$. In particular, A may be the energy. Let us calculate predictions concerning the observable A. If the transitory motions are neglected, we take only the quantized motions into consideration. Suppose we have a space (R) with at least as many dimensions as the spectrum of A has spectral values (taking account of their multiplicities). It is then a space of enumerably infinite dimension. To every observable A we can associate a linear operator $\mathfrak{A}_A$, which is a linear operator of the space (R), (whose spectrum is the spectrum of A, and such that the eigenvectors φ_i of $\mathfrak{A}_A$ form a complete basis:

$$\mathfrak{A}_A \varphi_i = a_i \varphi_i .$$

Comparing then with the general theory of predictions ([18]), we see that the φ_i may be taken as sure prediction elements for the observable A.

In particular, for the energy E we have a certain operator $\mathfrak{H}$, and if the system S is so that $\mathfrak{H}$ is a first integral to the classical approximation, this property must continue to hold. The energy operator $\mathfrak{H}$ is a first integral. This condition, and those preceeding, lead to a representation in a separable function space. Then, on account of the theorems of the general theory of predictions, the following condition holds: every prediction element X is de-

([18]) See Ref. ([4]).

velopable using a basis, that is

$$X = \sum_i c_i \varphi_i .$$

Then we have a vectorial representation, i.e. a representation of the same type as that of the usual wave mechanics. To this approximation, the functional character of the particle does not appear; we can limit ourselves to the pointwise description given by $M - \operatorname{sing} u$, and we can take M as argument of the elements of prediction, that is

$$X(u, t) \sim \Psi(\operatorname{sing} u, t) = \Psi(M, t) .$$

If, besides, we state that the spectra are strictly the same as in wave mechanics, we obtain exactly the same mechanics, if the spectra are slightly different, it amounts to modifying the form of the potentials or of the operators associated with the observables, but the theory remains a wave mechanics in the usual sense. Then we obtain this result.

If, in the functional theory of particles, we take into consideration only the quantized motions, this theory reduces to a wave mechanics (in the usual predictional form). This result involves the following consequences:

1) If we do not want to take into consideration transitory motions, the usual formalism of wave mechanics is sufficient, it is needless to use a functional theory (in particular the double solution theory).

2) As the present quantum theories are insufficient to describe a number of phenomena, a new theory becomes necessary; in order to avoid the usual quantum frame, this leads us to take into consideration transitory motions, and, hence, a concept of the quantization which will conform with that just developed.

Then, on the one hand, such a theory must conform with the usual wave mechanics which must appear as a limiting case, when only the quantized motions are taken into consideration; and, on the other hand, this theory must admit, to the pointwise and geometric optical approximation, a description in a pointwise mechanics with supplementary forces, introducing the indicator function of the spectrum as above. The functional theory of particles gives a general frame in which such a theory can be constructed.

If we use the predictional representation with functionals $X(u, t)$, for observables which are expressed uniquely by means of the quantized motions as the energy of quantized levels, we have degenerate observables giving a rough description of the system, and observables which are more sharp if we take into consideration transitory motions.

We may also use objectivistic representations where the particle moves

according to a determined wave u, to which we can associate a fluid. This permits us to represent the particle as a globule of fluid or as a moving point in a field on which it reacts.

Then it seems that, in order to obtain a new theory, it is first necessary to change the conception of quantized motions, considering them as motions having a stationary property among other motions, and no more as the only acceptable motions. Naturally, this change must satisfy the same experimental requirements as wave mechanics, and this demands a functional theory of particles ([19]).

* * *

In closing, I wish to express my sincere thanks to my colleagues Bopp (Munich), Čap, Gröbner, March (Innsbruck) for the fruitful discussions we have had on this subject.

([19]) More details and some properties of Q will be given in *La quantification en théorie fonctionnelle des corpuscules* (in press) [added in proofs].

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Note on a Surprising Result in Field-Mechanics.

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(ricevuto il 2 Marzo 1956)

Some years ago the author has investigated the classical equations of motion of a charged particle which result if Mawxell's equations are modified in a certain way (1). The term « field mechanics » was used for these equations of motion. According to recent discussions with Mr. Destouches, Paris, and Mr. F. Čap, Innsbruck (2) it may be of interest to know that the existence of absorptionless orbits is a general feature of field mechanics. I am stating the results here as mathematical facts without intending, for the moment, to draw any physical conclusions.

Just like in Born's theory (3) we have considered modified field equations of Maxwell type, distinguishing between the field strengths $(\mathfrak{E}, \mathfrak{B}) = (f_{\mu\nu})$ and $(\mathfrak{D}\mathfrak{H}) = (F_{\mu\nu})$. The main difference to Born's theory consists in the relation between the two field tensors, which is in our case a linear one:

$$F_{\mu\nu} = \int \varepsilon(x - x') f_{\mu\nu}(x') \, \mathrm{d}^4 x' \,. \tag{1}$$

The kernel $\varepsilon(x - x')$ of this integral transformation is a Lorentzinvariant function of the four-vector $x - x'$.

The restriction to linear field equations is by no means essential in the present context. In fact, we shall later on also consider non linear equations.

The derivation of the equations of motion of a charged point particle within

(1) F. Bopp: *Ann. Phys.*, **38**, 345 (1940); **42**, 573 (1942); l. c. (4).

(2) I am referring to the papers of Čap and Destouches in this issue.

(3) M. Born: *Proc. Roy. Soc.*, **143**, 410 (1934); M. Born and L. Infeld: *Proc. Roy. Soc.*, **144**, 425 (1934); **147**, 522 (1934); **150**, 141 (1935).

the frame of this generalized linear electrodynamics has been previously given [4]. If we make a non relativistic approximation, but do not neglect any retardation terms the result is

$$m_0 M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\dot{\mathfrak{v}} = \mathfrak{K} \ . \tag{2}$$

Here $M(\mathrm{d}/\mathrm{d}t)$ is an operator defined by

$$M(p) = 3\lambda_0\left\{\breve{g}(p) - \frac{1}{p^2}\int_0^p \breve{g}(\xi)\xi\,\mathrm{d}\xi\right\}, \tag{3}$$

where $\breve{g}$ is derived from

$$g(x) = \frac{c}{8\pi^3}\int \frac{\exp\left[ik_\mu x_\mu\right]}{k_\varrho^2\cdot\tilde{\varepsilon}(k)}\,\mathrm{d}^4k \ , \tag{4}$$

by a Laplacian transformation and $\tilde{\varepsilon}(k)$ is the Fourier transform of $\varepsilon(x)$.

In order to get restrictions for the arbitrary function I have formerly discussed various more or less necessary conditions, namely:

1) The electrical potential of the point charge at rest should be asymptotically equal to the Coulomb law.
2) The energy of the point charge should be finite.
3) The modified Maxwell equations should give no other quanta than photons.
4) The radiation forces should be the same as in Maxwell-Lorentz's theory.
5) There should be no run-away solutions.

The conditions 1), 2) and 5) cannot be abandoned. Condition 3) should hold at least in the non relativistic case. Obviously condition 4) is the weakest one. There is no reason for it except convenience.

Regarding the problem of Čap and Destouches it is reasonable to drop this condition or to replace it by another one. Therefore we don't make any further assumptions on the odd part of the function $M(p)$ in eq. (3), which gives the radiation damping according to the definition of the radiation field in Dirac's theory of electron [5]. Consequently we put now:

$$M(p) = M_1(p) - \frac{2e^2}{3c^3}M_2(p)p \ , \tag{5}$$

[4] F. Bopp: *Zeits. Naturfor.*, **1**a, 53 (1946); H. Hönl: *Ergeb. d. exakt. Naturw.*, **26**, 291 (1952).

where both M_1 and M_2 are even functions of p and equal to 1 if $p = 0$, and we get as an expression for the radiation force:

$$\mathfrak{K}_r = -\frac{2e^2}{3c^3} M_2 \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \frac{\mathrm{d}^2\mathfrak{v}}{\mathrm{d}t^2} . \tag{6}$$

The special case $M_2 = 1$ corresponds to the ordinary Lorentz force.

If M_2 is different from unity, the question arises: Do closed orbits exist on which the work done by $\mathfrak{K}_r$ will vanish? This will occur, if there are special orbits for which

$$\oint \mathfrak{K}_r \dot{\mathfrak{v}} \, \mathrm{d}t = 0 . \tag{7}$$

If we take the usual damping given by $\mathfrak{K}_r = (2e^2/3c^3)\ddot{\mathfrak{v}}$, this can only happen, if $\dot{\mathfrak{v}} = 0$, because

$$\oint \mathfrak{K}_r \mathfrak{v} \, \mathrm{d}t = \frac{2e^2}{3c^3} \oint \ddot{\mathfrak{v}} \mathfrak{v} \, \mathrm{d}t = -\frac{2e^2}{3c^3} \oint \dot{\mathfrak{v}}^2 \mathrm{d}t \leqslant 0 .$$

But in other cases eq. (7) has non trivial solutions giving orbits without radiation damping.

WILDERMUTH [6] has pointed out, how this is possible in the case of the unchanged Maxwell equations. Two conditions must come together. First the particle must have a finite extension, so that one part of it can absorb, what another part has emitted. Secondly outward emissions of the different parts must annihilate each other by interference.

In our case we have point charges; only the free charge (defined by div $\mathfrak{E}$ instead of div $\mathfrak{D}$) is spread in space [7]. Therefore the particle can reabsorb, what it has emitted, only if wave velocities less than that of light come in. This happens indeed and implies then that we have in general certain orbits without radiating damping.

A special equation of the announced type may be the following (*):

$$m\ddot{\mathfrak{r}} = -\frac{e^2}{r^2}\frac{\mathfrak{r}}{r} + \frac{2e^2}{3c^3} \prod_{n=1}^{\infty} \left(1 + \frac{\omega_0^2}{n^6(\mathrm{d}^2/\mathrm{d}t^2)}\right) \dddot{\mathfrak{r}} . \tag{8}$$

(*) The operator on the right hand side is equivalent to the integral operator:

$$\frac{1}{2\pi} \int \prod \left(1 - \frac{\omega_0^2}{n^6\omega^2}\right) \exp\left[i\omega(t - t')\right] \mathrm{d}\omega .$$

(5) P. A. M. DIRAC: *Proc. Roy. Soc.*, **167**, 148 (1938).
(6) K. WILDERMUTH: *Zeits. f. Naturfor.*, **10**a, 450 (1955).
(7) L. c. (4) and (3) too.

In this case radiation damping will vanish, if for instance the orbits are circles and the angular velocity is equal to

$$\omega = \frac{\omega_0}{n^3}, \tag{9}$$

which will give the Bohr orbits of the hydrogen atom if we put:

$$\omega_0 = \frac{me^4}{\hbar^3} = \frac{\alpha^2 mc^2}{\hbar}. \tag{10}$$

Obviously the above choice of the radiation reaction is quite haphazardous and will give senseless results in other cases. But one can try to construct the expression for the radiation reaction in such a way that it will yield in an arbitrary field of force as radiationsless orbits just those singled out by the quantum conditions [8]

$$I = \oint p \, \mathrm{d}q = 2\pi\hbar(n + \tfrac{1}{2}).$$

Then, starting from arbitrary initial conditions the particle will emit or absorb radiation until it settles down on a stationary orbit. We illustrate the pro cedure for the case of an oscillator and introduce action and angular variables I and $\varphi/2\pi$ (as judged from the approximation in which the radiation force is neglected altogether). In these variables the equations of motion including the radiation force should read

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\gamma f(I), \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega = \text{const}, \tag{12}$$

in which

$$I = \frac{\pi}{M\omega}(p^2 + M^2\omega^2 q^2) = \frac{2\pi}{\omega} H(p, q) \tag{13}$$

and

$$f(I) \geqslant 0, \quad f(I) = 0 \quad \text{if} \quad I = 2\pi\hbar(n + \tfrac{1}{2}). \tag{14}$$

In the case of an anharmonic oscillator only the relation (13) between the action variable I and the Hamiltonian must be changed.

Radiation damping is now a nonlinear expression like in the case proposed

[8] We use here the form given by E. C. KEMBLE: *The Fundamental Principles of Quantum Mechanics* (New York and London, 1937).

by ČAP. Field mechanics will give such expressions if we neglect retardation terms but do not drop the non-linear terms in velocity. They come in too, if we start with nonlinear field equations.

We conclude: The assumptions made by ČAP and DESTOUCHES may be realized by equations of motions derived from classical field theories. Modified Maxwell theories will give examples. Certainly we are not sure, that the field mechanical orbits have anything to do with quantum mechanics. But changing field theories is a necessity rather generally admitted. Nearly nothing is known, how to change it. In any case, if we consider modified theories, we must recognize, that orbits without radiation easily come in, and the question arises not yet answered to day: What—if any—is the relation between those orbits and quantum states?

SUPPLEMENTO AL VOLUME III, SERIE X
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Phaseshift Analysis of Pion-Nucleon Scattering Experiments. I. - General Outline.

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Contents. — Introduction. – **1.** General outline of the phasehift analysis problem. – **2.** The tracks for positive pion scattering. Anderson-Bethe and Fermi-Metropolis solutions. – **3.** The tracks for the scattering in the $T = \frac{1}{2}$ state. Steinberger solutions. – **4.** *D*-wave contribution in positive pion-nucleon scattering.

Introduction.

In recent years several efforts have been devoted to the phaseshift analysis of the available data concerning the scattering of positive and negative pions by protons, assuming explicitely the charge independence of nuclear forces and limiting the analysis to S and P wave contributions. The importance of this kind of analysis lies in the fact that the knowledge of the experimental phaseshifts limits the choice of the possible pion-nucleon interactions, and can suggest a theoretical picture of the pion-nucleon scattering in agreement with the experimental evidence.

Investigations performed both with the aid of electronic computers and graphical methods have so far shown that the existing experiments are, within the data uncertainties, compatible with the charge independence hypothesis, and have succeded in finding out, especially for the isotopic spin state $\frac{3}{2}$, the main features of the phaseshift behavior versus pion energy. Although the analysis in S and P waves only is not strictly correct, the good fit obtained for the angular distributions of the scattered pions has shown a posteriori that

this assumption may realistically be regarded as valid at least up to an energy of about 220 MeV. These two basic assumptions, i.e. the charge independence of nuclear forces and the S and P wave approximation, will be retained throughout the following discussion.

The electronic computers have been so far the main tool in the phaseshift analysis of experimental data. To achieve a better understanding of the phaseshift problem, we believe, however, to be of some interest the different line of attack presented in this paper, which is entirely based on the use of a linkage system, suitable to investigate one spin channel fermion-boson scattering processes. The impulse to find a new method for carrying out the analysis grew up because of the complexity of calculations, in particular for the scattering of negative pions, involved by the analytical procedure outlined in a previous paper (1). The mechanical analyzer was found to be the simplest device for a quick production of numerical results and, due to the mathematical symmetry between the relations defining the phaseshifts for $\frac{3}{2}$ and $\frac{1}{2}$ isotopic spin states, following from the charge independence hypothesis and valid to any order of partial waves, it can be used for both states.

After a general outline of the phaseshift analysis problem, given in Sect. 1, the behaviour of the phaseshifts versus pion energy is discussed in Sect. 2 for isotopic spin $\frac{3}{2}$ and in Sect. 3 for isotopic spin $\frac{1}{2}$, with reference to the causality prescriptions of the scattering theory. The extension of the method for the determination of D waves is given in Sect. 4. In the accompanying paper the application of the method will be discussed from the operational point of view, and illustrated in detail for the particular case of 189 MeV scattering.

1. – General Outline of the Phaseshift Analysis Problem.

The method generally used to determine the pion-nucleon phaseshifts consists either to explore the six dimensional phaseshift space for the minimum or minima of the function $M(\alpha_1, \alpha_{11}, \alpha_{13}; \alpha_3, \alpha_{31}, \alpha_{33})$ (2), or, starting from an initial set of six phaseshifts, to analyze the negative scattering data, asking the electronic computer to minimize the least squares quantity $M(A_-, B_-, C_-, A_0, B_0, C_0)$ (3), and then using the positive scattering data as a selection criterion for the solutions which meet the minimum condition.

It became clear recently (4) that not all the mathematical solutions so

(1) E. Clementel, G. Poiani and C. Villi: *Nuovo Cimento*, **2**, 389 (1955).

(2) E. Fermi, N. Metropolis and E. F. Alei: *Phys. Rev.*, **95**, 1581 (1954).

(3) F. de Hoffmann, N. Metropolis, E. F. Alei and H. A. Bethe: *Phys. Rev.*, **95**, 1586 (1954).

(4) H. L. Anderson, W. C. Davidon and U. E. Kruse: *Phys. Rev.*, **100**, 339 (1954).

obtained are also physical solutions, and a further selection is required. The selection criterion is provided by the causality principle. The method we shall use is based on the mechanical translation of the phaseshift equations ([5]), and in the exploration of a certain region in the (A_i, B_i, C_i)-space, where, for a given energy, « causal » solutions are to be found. Only in the final stage of the analysis we shall calculate the least squares sums in order to test the accuracy of the sets thus determined in fitting the experimental data ([6]).

In our method we need the coefficients A_i, B_i, C_i, describing in the S and P waves approximation the angular distribution according to

$$(1) \qquad k^2\sigma_i(\vartheta) = \left|\frac{\boldsymbol{a}_i}{2i} + \frac{\boldsymbol{b}_i}{2i}\cos\vartheta\right|^2 + \left|\frac{\boldsymbol{c}_i}{2i}\sin\vartheta\right|^2 = A_i + B_i\cos\vartheta + C_i\cos^2\vartheta\,,$$

where $i = 2T$ is twice the total isotopic spin. The complex quantities $\boldsymbol{a}_i$, $\boldsymbol{b}_i$, $\boldsymbol{c}_i$ are given by

$$(2) \qquad \begin{cases} \boldsymbol{a}_i = \exp[2i\alpha_i] - 1, \\ \boldsymbol{b}_i = 2\exp[2i\alpha_{i3}] + \exp[2i\alpha_{i1}] - 3, \\ \boldsymbol{c}_i = \exp[2i\alpha_{i3}] - \exp[2i\alpha_{i1}]\,, \end{cases}$$

and the coefficients of the angular distribution are expressed in terms of the complex quantities (2) by the following relations

$$(3) \qquad A_i = \tfrac{1}{4}(|\boldsymbol{a}_i|^2 + |\boldsymbol{c}_i|^2), \quad B_i = \tfrac{1}{2}\,\mathrm{Re}\,(\boldsymbol{a}_i\boldsymbol{b}_i^*), \quad C_i = \tfrac{1}{4}(|\boldsymbol{b}_i|^2 - |\boldsymbol{c}_i|^2)\,.$$

For isotopic spin $\frac{3}{2}$ we have

$$(4) \qquad A_3 = A_+, \qquad B_3 = B_+, \qquad C_3 = C_+\,,$$

and therefore $\sigma_3(\vartheta) = \sigma_+(\vartheta)$ gives the observed angular distribution, being A_+, B_+ and C_+ the least squares fit parameters of the experimental data. For the scattering in the state of isotopic spin $\frac{1}{2}$ the « auxiliary » differential cross-section $\sigma_1(\vartheta)$ has not a direct physical meaning, because the numbers A_1, B_1, C_1 are related to the primary experimental data A_-, B_-, C_-, A_0, B_0, C_0, describing the elastic respectively charge-exchange scattering of negative pions, and to A_+, B_+, C_+ by the following linear combinations

$$(5a) \qquad A_1 = \tfrac{1}{2}[3(A_- + A_0) - A_+]\,,$$

$$(5b) \qquad B_1 = \tfrac{1}{2}[3(B_- + B_0) - B_+]\,,$$

$$(5c) \qquad C_1 = \tfrac{1}{2}[3(C_- + C_0) - C_+]\,,$$

([5]) E. CLEMENTEL and C. VILLI: *Nuovo Cimento*, **2**, 845 (1955).

([6]) I. GABRIELLI, G. IERNETTI, E. CLEMENTEL and C. VILLI: *Suppl. al Nuovo Cimento*, **3**, 496 (1956).

and therefore the auxiliary differential cross section is fixed by

$$\sigma_1(\vartheta) = \tfrac{3}{2}[\sigma_-(\vartheta) + \sigma_0(\vartheta)] - \tfrac{1}{2}\sigma_+(\vartheta) . \tag{6}$$

From Eqs. (5) or (6), obtained by elementary but tedious manipulations using the relations listed in the Appendix (7) of ref. (1), it follows that

$$\boldsymbol{a}_1 = \tfrac{1}{2}(\boldsymbol{a}_- - \boldsymbol{a}_+), \quad \boldsymbol{b}_1 = \tfrac{1}{2}(\boldsymbol{b}_- - \boldsymbol{b}_+), \quad \boldsymbol{c}_1 = \tfrac{1}{2}(\boldsymbol{c}_- - \boldsymbol{c}_+) . \tag{7}$$

With this formalism, it can be easily seen (5) that the two sets of equations leading to the phaseshifts α_i, α_{i1}, α_{i3} ($i = 1, 3$) have the same mathematical structure. The time consuming parametrization of α_{i1} and α_{i3} as functions of α_i, involved by the analytical method (1), can be avoided by resorting to the kinematical translation of the following pion-nucleon phaseshift equations

$$\cos 2\alpha_i + 2\cos 2\alpha_{i3} + \cos 2\alpha_{i1} = u_i(k) , \tag{8a}$$

$$\sin 2\alpha_i + 2\sin 2\alpha_{i3} + \sin 2\alpha_{i1} = v_i(k) , \tag{8b}$$

$$\cos 2\alpha_i + \cos 2(\alpha_{i3} - \alpha_{i1}) = w_i(k) , \tag{8c}$$

where the right-hand side quantities are defined according to

$$u_i(k) = 4 - (k^2/2\pi)\sigma_i(k) = 4 - 2(A_i + C_i/3) , \tag{9a}$$

$$v_i(k) = \pm 2k[\sigma_i(0^0) - (k/4\pi)^2\sigma_i^2(k)]^{\frac{1}{2}} = \pm 2[(A_i + B_i + C_i) - (A_i + C_i/3)^2]^{\frac{1}{2}} , \tag{9b}$$

$$w_i(k) = 2[1 - k^2\sigma_i(90^0)] = 2(1 - A_i) . \tag{9c}$$

The first and second of Eqs. (8) have been derived applying to the differential cross-section (1) the following theorems

$$k\sigma_i(k) = 4\pi \operatorname{Im} f_i(k) , \tag{10a}$$

$$\sigma_i(0^0) = [\operatorname{Re} f_i(k)]^2 + (k/4\pi)^2\sigma_i^2(k) , \tag{10b}$$

where

$$f_1(k) = \tfrac{1}{2}[3f_-(k) - f_+(k)], \quad f_3(k) = f_+(k) , \tag{11}$$

are the forward scattering amplitudes. The third of Eqs. (8) follows from the first of Eqs. (3), and plays the role of freezing condition for the linkage

(7) In the right-hand side of the first line of Eqs. (A.2) and (A.3) a factor (1,9) respectively (2/9) is missing.

system, fixing that configuration of the mechanical analyzer, which corresponds to the required solutions. For further details we refer to the paper quoted in ref. (5).

Squaring and adding the first two of Eqs. (8), and taking account of Eq. (8c), the S-wave phaseshift for both isotopic spin states can be expressed as follows

$$\cos(2\alpha_i - \varphi) = \pm \frac{u_i^2 + v_i^2 + 4(1 + w_i)}{2[u_i^2 + v_i^2 + 4(1 - u_i)]^{\frac{1}{2}}}, \tag{12a}$$

where

$$\operatorname{tg}\varphi = \frac{v_i}{u_i - 2}. \tag{12b}$$

Once the phaseshift α_i is known, the two P-phaseshifts follow immediately from Eqs. (a, b) (see Eq. (4) and (5) of ref. (1)). We have purposely neglected this kind of approach because it gives rise to ambiguities both in sign and value of all the phaseshifts, the elimination of which requires tedious calculations, as in the parametrization method already suggested (1).

It is interesting to observe that the differential cross-section (1) can be obtained from the general expression

$$k^2\sigma_i(\vartheta) = \left| \sum_{l=0}^{\infty} \left\{ (l+1)\exp[i\alpha^l_{i,2l+1}]\cdot\sin\alpha^l_{i,2l+1} + l\exp[i\alpha^l_{i,2l-1}] \right\} P_l(\cos\vartheta) \right|^2 + \tag{13}$$

$$+ \left| \sum_{l=0}^{\infty} \left\{ \exp[i\alpha^l_{i,2l+1}]\cdot\sin\alpha^l_{i,2l+1} - \exp[i\alpha^l_{i,2l-1}]\sin 2\alpha^l_{i,2l-1} \right\} P^1_l(\cos\vartheta) \right|^2,$$

limiting the expansions to $l = 1$ and using the notation (2), where the superscript l in the phaseshift symbols has been dropped and, for $l = 0$, we have put $\alpha_i = \alpha^0_{i1}$. For $T = \frac{3}{2}$, Eq. (13) is the obvious generalization of Bloch's (6) formula for one spin-channel fermion-boson scattering. For $T = \frac{1}{2}$, the analogy is only formal.

The main advantage of the mechanical analyzer is the visualization of the phaseshift continuity versus pion energy, and the possibility to carry out a straightforward test of their stability against variations of the angular distribution coefficients. This procedure allows to follow every stage of the analysis keeping sight simultaneously both of the experimental data and the mathematics concerned with the phaseshift equations. The considerable insight into the structure of the phase fitting problem, obtained by means

(6) F. BLOCH: *Phys. Rev.*, **58**, 829 (1940).

of the Ashkin and Vosko (9) graphical method, is here increased by the introduction of a new element, i.e. the mouvement.

2. – The Tracks for Positive Pion Scattering. Anderson-Bethe and Fermi-Metropolis Solutions.

The sign ambiguity of $v_3(k)$, expressing the well known fact that the phaseshift equations are unaltered by a simultaneous reversal of the sign of all the phaseshifts, introduces solutions which are physically unacceptable, because they do not satisfy the causality condition, requiring the scattered wave to be zero until the incident wave has not reached the scatterer. When this condition is consistently taken into account in the pion-nucleon scattering theory (10), it leads to a relation between the real part of the forward scattering amplitude and the total scattering cross section. By using this integral relation (4) and extrapolating to zero energy the S-wave phaseshift according to (11) $\alpha_3 = -0.11\eta$, it can be established that Re $f_3(k)$, starting from zero value, is negative up to about 40 MeV, then positive up to about 180 MeV, where it changes again its sign, decreasing then rapidly with increasing energy. Because of the relation

$$v_3(k) = 2k \text{ Re } f_3(k) , \tag{14}$$

one can calculate the behaviour of Re $f_3(k)$ with different known sets of phaseshifts. The important conclusion follows that α_3 must be negative. As it is well known, the positive sign of $v_3(k)$ in the region of 110 MeV can be inferred also from the sign of the nuclear-Coulomb interference, which has been found negative (11). In fact, using van Hove's (12) expression for the differential cross-section, and neglecting $e^2/\hbar v$ compared to 1, a destructive Coulomb interference requires a positive sign for the quantity Im $(\boldsymbol{a}_+)$ – Im $(\boldsymbol{b}_+)$ cos ϑ, which at small angles (cos $\vartheta \simeq 1$) is practically $v_3(k)$. According to this argument, below 40 MeV and beyond 180 MeV, due to the negative sign of $v_3(k)$, the Coulomb interference is expected to be constructive. Such an effect has been detected recently at 216 MeV (13).

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(9) J. Ashkin and H. S. Vosko: *Phys. Rev.*, **91**, 1248 (1953).

(10) R. Karplus and M. A. Ruderman: *Phys. Rev.*, **98**, 771 (1955); M. Goldberger, H. Miyazawa and R. Oehme: *Phys. Rev.*, **99**, 986 (1955).

(11) J. Orear: *Phys. Rev.*, **96**, 176 (1954). The behaviour of the phaseshifts α_1 and α_3 in the limit $k > 0$ is in agreement with a recent unpublished result of M. C. Rinehart, K. C. Rogers and L. M. Lederman, who give $\alpha_1 = (+0.17 \pm 0.04)\eta$ and $\alpha_3 = (-(0.10 \pm 0.02)\eta$, where η is the relative momentum in units μc.

(12) L. van Hove: *Phys. Rev.* **88**, 1358 (1952).

(13) H. D. Taft: *Phys. Rev.*, **101**, 1116 (1956).

In Fig. 1 we have drawn with a continuous line the causal track, to be described by the terminal point P_1 of the analyzer with varying pion energy. We shall call this track as P-track, and the set of phaseshifts obtained moving along it as Anderson-Bethe solutions. The dotted line is the mirror track of the causal one. If one moves along the P-track up to the second inversion point, and then proceedes along the (non-causal) dotted track, one obtains the so called Fermi-Metropolis solutions. The resonant behaviour of the phaseshifts α_{33} is found only moving along the P-track.

It is important, expecially from the point of view of the phaseshift continuity, to establish some general relations, which are satisfied by the Anderson-Bethe solutions. The most remarkable fact is that the so called Yang set of P-phaseshifts varies much faster than the Fermi one with increasing pion energy ([1]), so that for an energy somewhere between 170 and 180 MeV we have $\alpha_{33}^{(+)} = \alpha_{33}^{(-)}$. According to Eq. (4) of ref. ([1]), this equality implies $\lambda_3^2(k) = \mu_3(k)$. This relation, verified at the crossing energy of Fermi and Yang solutions, is very critic, since it requires the curve $\mu_3(k)$ to be tangent to the curve giving $\lambda_3^2(k)$, because for all energies it must be $\lambda_3^2(k) \geqslant \mu_3(k)$, otherwise the P-state phaseshifts would become complex. The existence of this crossing energy follows from the fact that the Fermi $\alpha_{31}^{(-)}$ phaseshift remains small and negative throughout the energy region. This crossing-over of the two phaseshifts, which is clearly shown by the kinematical analysis, is difficult to realize by a static analysis performed with the aid of graphical methods.

It is preferable to locate the inversion energy in terms of the coefficients of the angular distribution using Eqs. (8). The inversion energy is found by satisfying the following relation

$$[-w_3(w_3+2)]^{\frac{1}{2}} + [(u_3-w_3)(2-u_3-w_3)]^{\frac{1}{2}} = v_3 . \tag{15}$$

Similarly, the resonance energy for $\alpha_{33}^{(-)}$ is given by

$$[-(u_3+w_3)(4+u_3+w_3)]^{\frac{1}{2}} + [(w_3-u_3)(4+u_3-w_3)]^{\frac{1}{2}} = 2v_3 . \tag{16}$$

Once these energies have been determined on the basis of the experimental coefficients, the phaseshifts at the inversion energy are given by

$$\cos 2\alpha_3 = 1 + 2A_3, \quad \cos 2\alpha_{33}^{(\pm)} = 1 - \tfrac{2}{9}C_3, \quad 2\alpha_{31}^{(\pm)} = 2\alpha_{31}^{(\pm)} \pm \pi , \tag{17}$$

while at the resonance energy they follow from the relations

$$\cos 2\alpha_3 = 4 - 2A_3 + \tfrac{1}{2}C_3, \quad 2\alpha_{33}^{(-)} = \pi, \quad \cos 2\alpha_{31}^{(-)} = 2 - \tfrac{1}{3}C_3 . \tag{18}$$

In a similar way it is possible to fix in terms of the experimental coeffi-

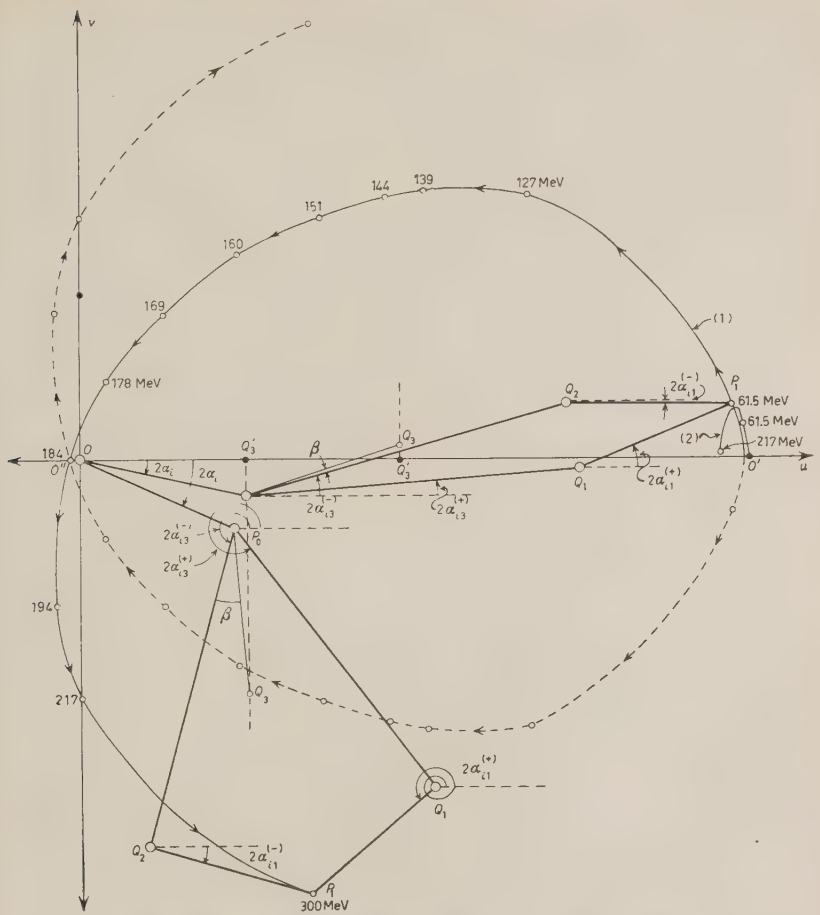

Fig. 1. – Linkage system for pion-nucleon scattering in S and P waves: $OP_0=P_0Q_3=Q_2P_1=Q_1P_1=1$, $P_0Q_2=P_0Q_1=2$, $OO'=4$. The coordinates of the point P_1 are $u_i(k)$ and $v_i(k)$; the input parameter $w_i(k)$ is given by OQ_3'. Solutions labelled with (+) and (—) are Yang and Fermi solutions for isotopic spin $T=\frac{1}{2}$ $(i=1, 3)$. The full line curves (1) and (2) show the causal track of the terminal P_1 for the scattering in the isotopic spin $\frac{3}{2}$ respectively $\frac{1}{2}$ for pion energies varying from 40 to 300 MeV. The two illustrative causal tracks have been evaluated using the angular distribution coefficients listed in Table XVI of ref. [3]. The point at 300 MeV has been evaluated using Margulies data (*Bull. Am. Phys. Soc.*, **30**, 73 (1955)). It is seen that the region of the (u, v)-plane close to the point O' belongs either to positive scattering below approximately 70 MeV or to negative scattering up to about 200 MeV. The Fermi-Metropolis solutions are non causal only for scattering energies higher than that satisfying the equation $v_3(k)=0$. These non causal solutions are found by moving the terminal P_1 upwards along the dotted track, starting from the point O''.

cients the energies at which the Yang phaseshift $\alpha_{31}^{(+)}$ reaches the values $\pi/2$ or π. Since the phaseshift $\alpha_{31}^{(+)}$ keeps increasing beyond the energy at which it assumes the value π, the continuity of Yang solutions requires the ambiguity relation

$$\alpha_{33}^{(+)} - \alpha_{31}^{(+)} = \alpha_{31}^{(-)} - \alpha_{33}^{(-)} , \tag{19}$$

characteristic of single channel fermion-boson scattering ([14]), to be verified below and above the resonance energy.

Finally, since any point on the P-track has to satisfy the inequality

$$[(u_3 - \cos 2\alpha_3)^2 + (v_3 - \sin 2\alpha_3)^2]^{\frac{1}{2}} \geqslant 1 , \tag{20}$$

we see that, within the circle $u_3^2(k) + v_3^2(k) = 1$ and $u_3(k) > 0$, positive values of α_3 are excluded, independently from causality considerations, for pion energies satisfying the inequality $2u_3(k) \leqslant u_3^2(k) + v_3^2(k)$. This happens in the energy interval $170 \div 180$ MeV. In terms of the angular distribution coefficients, the previous inequality reads $B_3 \geqslant 2 \cdot (A_3 - 1)$.

3. – The Tracks for the Scattering in the $T = \frac{1}{2}$ State. Steinberger Solutions.

While the positive pion scattering is a pure $T = \frac{3}{2}$ scattering, in the negative pion scattering both isotopic spin states are effective, and only considering the auxiliary cross-section (6) we obtain a system of equations like (8), which is the countepart of the same system for the $T = \frac{3}{2}$ state, insofar as it contains the $T = \frac{1}{2}$ phaseshifts only.

Applying the dispersion relations to the scattering of negative pions by protons, ANDERSON *et al.* ([4]) have shown that $\mathrm{Re}\, f_1(k)$ is positive all over the energy range, and therefore the sign of $v_1(k)$, because of

$$v_1(k) = 2k\, \mathrm{Re}\, f_1(k) , \tag{21}$$

is fixed. In this case the expression for $\mathrm{Re}\, f_1(k)$ contains a contribution due to the bound state proton $+ \pi^-$, which is essential to get agreement with the behaviour of $\mathrm{Re}\, f_1(k)$ calculated using in (8*b*) known sets of phaseshifts for the $T = \frac{1}{2}$ state. In extrapolating to zero energy the α_1 phaseshift, use has been made of Orear's relation ([11]) $\alpha_1 = 0.16\eta$. The bound state contribution contains the pion-nucleon interaction strength, and therefore the knowledge of the $T = \frac{1}{2}$ state phaseshifts gives indirectly, through Eq. (21), information about this important physical quantity.

([14]) E. CLEMENTEL and C. VILLI: *Nuovo Cimento*, **2**, 1121 (1955).

In Fig. 1 is shown with a continuous line the causal track for $T = \frac{1}{2}$ state scattering (curve 2), and we shall call Anderson-Bethe solutions the set of phaseshifts obtained moving along this track.

The knowledge of the quantity $r_1(k)$ gives information about the nuclear-Coulomb interference in the process $\pi^- \to \pi^-$. In fact, using the same approximation as in the process $\pi^+ \to \pi^+$, it is easy to establish that the sign of the nuclear-Coulomb interference is fixed by the sign of the quantity $\mathrm{Im}\,(\boldsymbol{a}_-) + \mathrm{Im}\,(\boldsymbol{b}_-)\cos\vartheta$, which for small angles ($\cos\vartheta \simeq 1$) can be written as $\mathrm{Im}\,(\boldsymbol{a}_- + \boldsymbol{b}_-) = 3r_-(k) = r_3(k) + 2r_1(k)$, being $r_-(k) = 2k\ \mathrm{Re}\, f_-(k)$. We can therefore conclude that, being $r_1(k)$ everywhere positive, when the positive pion nuclear-Coulomb interference is destructive, the negative pion one is certainly constructive, but the opposite may not be true.

We shall conventionally call Yang (+) and Fermi (−) solutions for isotopic spin $\frac{1}{2}$ those satisfying the P-state ambiguity

$$\alpha_{13}^{(+)} - \alpha_{11}^{(+)} = \alpha_{11}^{(-)} - \alpha_{13}^{(-)}. \tag{22}$$

It is clear that, in the charge independence hypothesis, there exists a priori a maximum number of four sets of six phaseshifts α_i, α_{i1}, α_{i3} ($i = 1, 3$) consistent with given values of A_i, B_i and C_i, obtained combining each of the two solutions for the $T = \frac{1}{2}$ state with the two for the $T = \frac{3}{2}$ state. It is difficult to decide which one out of the four sets is the correct one. In the positive pion scattering, due to the strong resonance in the $P_{\frac{3}{2}}$ state, the Fermi solution is likely to be the true one. In the negative pion scattering there is a fairly clear evidence of a maximum at about 1 GeV, of which the $T = \frac{1}{2}$ state should be responsible ([15]). Unfortunately no differential data are available sofar beyond 217 MeV, and the information below this limit on the behaviour of the phaseshifts for the $T = \frac{1}{2}$ state are rather confused.

From Fig. 1 it is seen that the input parameters $u_3(k)$ and $r_3(k)$ for positive pion scattering up to about 70 MeV are of the same order of magnitude as the input parameters $u_1(k)$ and $r_1(k)$ for the $T = \frac{1}{2}$ state scattering at the same energy. A very peculiar situation then arises, because the phaseshifts α_3, α_{31} and α_{33} become also of the same order of magnitude as the phaseshifts α_1, α_{11} and α_{13} respectively. Therefore they appear, so to speak, indistinguishable with respect to the isotopic spin state they belong to. This ambiguity in attributing the correct phaseshifts to each one of the isotopic spin states does not occur in our method, since their determination is carried out separately. However, if we use the graphical or mechanical translation of the negative

([15]) For an interpretation of this maximum in terms of a resonant pion-pion interaction see.: F. J. Dyson: *Phys. Rev.*, **99**, 1037 (1955) and G. Takeda: *Phys. Rev.*, **100**, 440 (1955).

scattering equations in the (u^*, v^*_-) plane, using the input parameters $u^*_-(k) = -3u_-(k) = u_3(k) + 2u_1(k)$ and $v^*_-(k) = 3v_-(k) = v_3(k) + 2v_1(k)$, as it is shown in Fig. 2 (which visualizes in some way the procedure followed in the analysis of negative pion scattering data with electronic computers), one immediately realizes that when the linkage system is nearly stretched along the u^* axis, as far as the fit of experimental data is concerned, the fit is not destroyed if the phaseshifts for the $T = \frac{1}{2}$ state are distributed among the degrees of freedom of the $T = \frac{3}{2}$ stage of the linkage system, and viceversa.

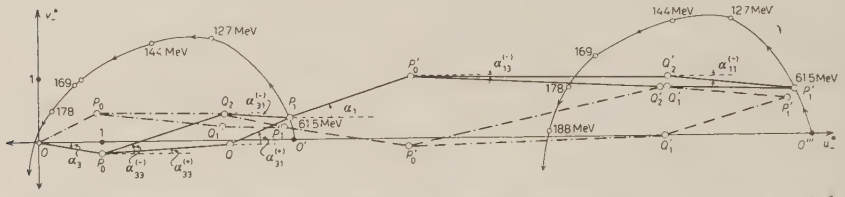

Fig. 2. – Linkage system operating with both S- and P-stages for isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$. $P_1P'_0 = Q'_1P'_1 - Q'_2P'_1 = 2$, $P'_0Q'_1 = P'_0Q'_2 = 4$. The coordinates of the terminal P'_1 are $u^*_-(k) = -u_3(k) + 2u_1(k)$ and $v^*_-(k) = v_3(k) + 2v_1(k)$. The configuration of the system is shown for the scattering energy of 61.5 MeV, in two positions corresponding to solutions I-II (full line) and III-IV (dotted line) of Table I. It is seen how at this energy the interchange of the phaseshifts for the two isotopic spin states does not alter appreciably the position of the « experimental » points P_1 and P'_1.

This ambiguity, of course, is peculiar for low energy scattering only. The set of phaseshifts with isotopic spin interchanged constitute the solutions known as Steinberger solutions, to be rejected lacking of continuity with higher energy phaseshifts. In fact, inspection of Table I ([16]) shows that if we exchange the isotopic spin index in the III and IV solutions at 61.5 MeV the

TABLE I. – *Phaseshift solutions in degrees at the scattering energy of* 61.5 MeV. Solutions I and II are Fermi and Yang solutions respectively; solutions III and IV are Steinberger solutions.

Solutions	α_3	α_1	α_{33}	α_{31}	α_{13}	α_{11}	M
I	— 5.4	+ 9.4	+ 8.3	— 1.8	— 2.0	+ 0.1	86
II	— 5.4	+ 9.4	+ 1,6	+ 11,6	— 0.7	— 2.7	86
III	+ 13.0	— 3.6	— 2.7	— 1.1	+ 6.3	— 1.2	143
IV	+ 13.0	— 3.6	— 1.6	— 3.1	+ 1.3	+ 9.0	143

([16]) This Table reproduces the first four solutions of the Table IV given in the paper of ref. ([2]). In this connection see also sec. 33*b* and 33*c* in the book of H. A. BETHE and F. DE HOFFMANN: *Mesons and Fields*, vol. 2 (New York, 1955).

misterious features of the so-called Steinberger solutions disappear, and what remains is nothing but a solution of the usual type corresponding to a value of M which is not the minimum.

The obvious consequence of this argument is that, when the scattering energy is such that the sets of phaseshifts for the two isotopic spin states are very different, the Steinberger solutions disappear, because no phaseshift interchange between the $T = \frac{3}{2}$ and $T = \frac{1}{2}$ state can possibly occur. This explains Orear's ([17]) failure to find solutions of Steinberger's type at an energy of 113 MeV. From the values of the phaseshifts given in Table I for the two Steinberger solutions (III and IV), it is seen that both $r_3(k)$ and $r_1(k)$ are positive: the solutions lie therefore on a causal track, and obviously no relation whatsoever exists between these and the Fermi-Metropolis solutions at higher energies, which lie on the non-causal track.

4. – D-Wave Contribution in Positive Pion-Nucleon Scattering.

The conclusions drawn so far from experiments on pion-nucleon scattering are based on the assumption that only S and P waves contribute to the scattering. As it has been noted before, this assumption seems to be true at least up to a pion energy of 220 MeV ([18]). Several Authors ([19]) have tried, on the basis of simplifying assumptions, to estimate D-wave contributions from the analysis of experimental data, but the accuracy of the experiments so far performed is not high enough to draw definite conclusions on small D-wave effects. Nevertheless, in view of high energy future experiments, we shall extend our method up to $l = 2$ waves, and give the correlation between the two D waves for positive pion-proton scattering in the energy region where the P-D interference term is predominant compared to the pure D-term.

When higher order waves are considered it is convenient to write Eq. (13) in the form

$$k^2\sigma_+(\vartheta) = \left| \sum_{l=0}^{\infty} \frac{\boldsymbol{a}_l}{2i} P_l(\cos\vartheta) \right|^2 + \left| \sum_{l=0}^{\infty} \frac{\boldsymbol{b}_l}{2i} P_l^1(\cos\vartheta) \right|^2 , \tag{23}$$

where we have defined ([20])

$$\boldsymbol{a}_l = (l+1)\exp[2i\alpha^l_{3,2l+1}] + l\exp[2i\alpha^l_{3,2l-1}] - (2l+1) , \tag{24a}$$

$$\boldsymbol{b}_l = \exp[2i\alpha^l_{3,2l+1}] - \exp[2i\alpha^l_{3,2l-1}] . \tag{24b}$$

([17]) J. Orear: *Phys. Rev.*, **96**, 1417 (1954).

([18]) M. O. Stern, J. Ashkin, J. P. Blaser and F. Feiner: *Bull. Am. Phys. Soc.*, **1**, 72 (1956).

([19]) J. Orear: *Bull. Am. Phys. Soc.*, **30**, n. 1 (1955); M. Glicksman: *Phys. Rev.*, **94**, 1335 (1954); R. S. Margulies: *Bull. Am. Phys. Soc.*, **30**, n. 7 (1954).

([20]) The correspondence with the notation of Sect. **1** is: $\boldsymbol{a}_+ = \boldsymbol{a}_0$, $\boldsymbol{b}_+ = \boldsymbol{a}_1$, $\boldsymbol{c}_+ = \boldsymbol{b}_1$.

A straightforward calculation shows that the differential cross-section (23,) up to $l_{\max} = 2$, can be written as ($n_{\max} = 2l_{\max}$)

$$k^2\sigma_+(\vartheta) = \sum_{n=0}^{4} A_n \cos^n \vartheta \,, \tag{25}$$

where the expansion coefficients are given by

$$4A_0 = |\boldsymbol{a}_0|^2 + |\boldsymbol{b}_1|^2 + \tfrac{1}{4}|\boldsymbol{a}_2|^2 - \mathrm{Re}\,(\boldsymbol{a}_0\boldsymbol{a}_2^*) \,, \tag{26a}$$

$$4A_1 = 2\,\mathrm{Re}\,(\boldsymbol{a}_0\boldsymbol{a}_1^*) - \mathrm{Re}\,(\boldsymbol{a}_1\boldsymbol{a}_2^*) + 6\,\mathrm{Re}\,(\boldsymbol{b}_1\boldsymbol{b}_z^*) \,, \tag{26b}$$

$$4A_2 = |\boldsymbol{a}_1|^2 - |\boldsymbol{b}_1|^2 - \tfrac{3}{2}|\boldsymbol{a}_2|^2 + 9|\boldsymbol{b}_2|^2 + 3\,\mathrm{Re}\,(\boldsymbol{a}_0\boldsymbol{a}_2) \,, \tag{26c}$$

$$4A_3 = 3\,\mathrm{Re}\,(\boldsymbol{a}_1\boldsymbol{a}_2^*) - 6\,\mathrm{Re}\,(\boldsymbol{b}_1\boldsymbol{b}_2^*) \,, \tag{26d}$$

$$4A_4 = \tfrac{9}{4}|\boldsymbol{a}_2|^2 - 9|\boldsymbol{b}_2|^2 \,. \tag{26e}$$

Applying now the two theorems (10*a*, *b*) to Eq. (23) and taking into account of Eq. (24), we obtain

$$\cos 2\alpha_3 + 2\cos 2\alpha_{33} + \cos 2\alpha_{31} + 3\cos 2\beta_{35} + 2\cos 2\beta_{33} = U_+(k) \,, \tag{27a}$$

$$\sin 2\alpha_3 + 2\sin 2\alpha_{33} + \sin 2\alpha_{31} + 3\sin 2\beta_{35} + 2\sin 2\beta_{33} = V_+(k) \,, \tag{27b}$$

where we have denoted with $\beta_{35} = \alpha_{35}^2$ and $\beta_{33} = \alpha_{33}^3$ the two D-phaseshifts, and introduced the quantities

$$U_+(k) = 9 - (k^2/2\pi)\sigma_+(k) - \sum_{j=0}^{l_{\max}} \left[(2j+1) - \frac{2A_{2j}}{2j+1}\right], \tag{28a}$$

$$V_+(k) = \pm 2k[\sigma_+(0^\circ) - (k/4\pi)^2\sigma_+^2(k)]^{\frac{1}{2}} = \pm\left[\sum_{j=0}^{2l_{\max}} A_j - \left(\sum_{l=0}^{l_{\max}} \frac{A_{2j}}{2j+1}\right)^2\right]^{\frac{1}{2}} . \tag{28b}$$

It is easy to prove that for any set (α_3, α_{31}, α_{33}) there exist two different sets of D-phaseshifts, characterized by the same value of the difference $|\beta_{35} - \beta_{33}|$. In fact calling $u'_+(k)$ and $v'_+(k)$ the sums defined in (8*a*, *b*) when D waves are effective, from Eqs. (27) we derive

$$12\cos 2(\beta_{35} - \beta_{33}) = (U_+ - u'_+)^2 + (V_+ - v'_+)^2 - 13 \,. \tag{29}$$

We have therefore a D-wave ambiguity for a given set of S and P phaseshifts, as we had a P-wave ambiguity for a given S-phaseshift.

From Eq. (26*e*) we can express β_{35} in terms of β_{33} through the relation

$$\cos 2\beta_{35} = \frac{1 + P \pm Q[Q^2 - P(2+P)]^{\frac{1}{2}}}{1+Q^2} \,, \tag{30}$$

where

$$P(\beta_{33}, A_4) = \frac{8A_4}{90\cos 2\beta_{33} - 135}, \tag{31a}$$

$$Q(\beta_{33}) = \frac{90\sin 2\beta_{33}}{90\cos 2\beta_{33} - 135}. \tag{31b}$$

Eq. (30) shows that the particular solution $\beta_{35}=0$ is mathematically possible for any value of β_{33} when $A_4=0$. Since the coefficient A_4 is expected to be very small for energies up to 300 MeV, we list in Table II the correlation between the two D phaseshifts expressed by Eq. (30), in the particular case $A_4 = 0$, for some values of the phaseshift β_{33}. Using the phaseshifts so correlated, we have calculated also the quantities $5-(U_-(k)-u'_-(k))$ and $V_+(k)-v'_+(k)$, which give in first approximation the shift in the (u, v) plane of the coordinates $u_+(k)$ and $v_+(k)$ which would be obtained considering S and P waves only. From the Table it is seen that « switching on » the D waves the P-track is shifted to the right in the (u, v) plane into a new P'-track (Fig. 3), which is described by the point P'_1 with coordinates $u'_-(k)$ and $v'_-(k)$. The point P_2 with coordinates $U_+(k)$ and $V_+(k)$ describes what we can call the D-track. The correlation between the P, P' and D-track is visualized in Fig. 3.

TABLE II. – *Correlation between the D-phaseshifts* β_{33} *and* β_{35} *(in degrees) assuming* $A_4=0$.

$\pm\beta_{33}$	0.25	0.5	0.75	1	2	3	4	5
$\mp\beta_{35}$	1	2	3	4	7.9	11.7	15.2	18.6
$5-u_D$	0.002	0.008	0.017	0.043	0.118	0.257	0.432	0.642
v_D $(\beta_{33}\lesssim 9)$	± 0.087	± 0.175	± 0.261	± 0.431	± 0.676	± 0.982	± 1.240	± 1.467

It is apparent from Eq. (30) that for $A_4=0$ the only possibility for the two D-phaseshifts to be equal is to be both zero. This circumstance does not seem to be reproduced by the Ross potential [21] in the case $A_4=0$ $(P=0)$.

The D-phaseshifts obtained from Eq. (30) show the same correlation trend as the D-phaseshifts calculated theoretically by NELKIN [22], and by HENLEY and RUDERMAN [23], who give for $T=\frac{3}{2}$ the correlation $\beta_{35} = -3.81\beta_{33}$.

It is interesting to observe that, denoting with $\beta_{33}^{(0)}$ the phaseshift belonging to the D-solution having $\beta_{35}=0$, the other solution, consistent with Eq. (29),

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[21] M. ROSS: *Phys. Rev.*, **95**, 1687 (1954).
[22] Result quoted by H. A. BETHE and F. DE HOFFMANN: op. cit., pag. 128.
[23] E. M. HENLEY and M. A. RUDERMAN: *Phys. Rev.*, **90**, 719 (1953).

must satisfy the following relation

$$\beta_{35} - \beta_{33} = \beta_{33}^{(0)} . \tag{32a}$$

It is clear from Eq. (32*a*) that β_{33} and β_{35} must have opposite signs. For instance, the solution proposed by MARGULIES [19] at 400 MeV, $\beta_{33}^{(0)} = -22°$, is associated also with the set $\beta_{33} = 4.6°$ and $\beta_{35} = -17.4°$.

In the general case ($A_4 = 0$), the ambiguity relation between the two D-phaseshifts is given by

$$\beta_{35} - \beta_{33} = \beta'_{33} - \beta'_{35} . \tag{32b}$$

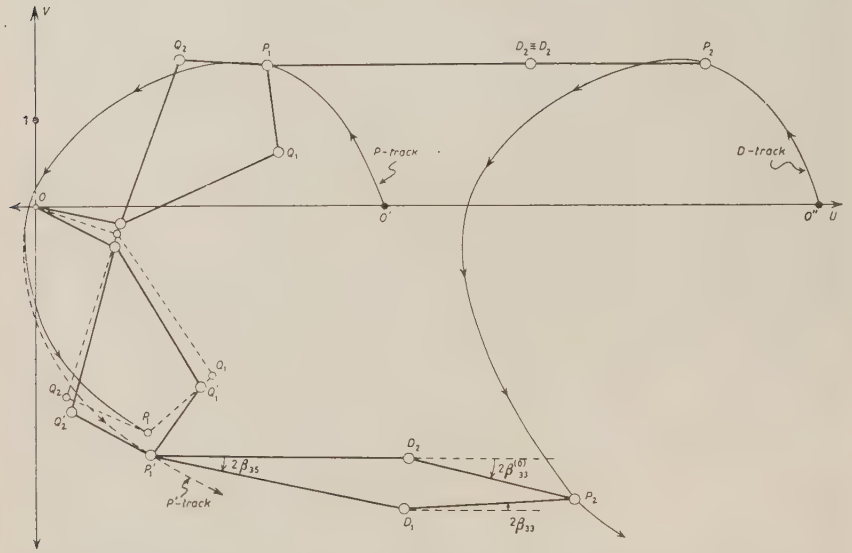

Fig. 3. – Linkage system for the determination of S, P and D phaseshifts in positive pion-nucleon scattering. The D-linkage $P_1' D_1 P_2 D_2$ is connected to the terminal P_1 of the mechanical system considered in Fig. 1 ($P_1' D_1 = P_2' D_2 = 3$, $D_1 P_2 = D_2 P_2 = 2$). The coordinates of the terminal P_2 are given by Eqs. (28*a*,*b*). As the pion energy increases the input terminal P_2 moves along the causal D-track (curve 1) driving the output terminal P_1 along the causal P-track. As long as the D-phaseshifts are zero or negligible the P and D tracks are connected by a rigid displacement, because the two branches of the D stage are stretched and superimposed. The analyzer shown in the upper half part of the (U, V)-plane refers to the scattering situation at about 120 MeV. As the contribution of D-waves becomes sensible, the point P_1 leaves the original P-track (full line) and proceeds along the perturbed P'-track (dotted line) and the D-stage opens. The given D-track ($A_4 = 0$) is only indicative, since the required experimental information is at the present not available.

Let us now consider the procedure to be followed in order to solve Eqs. (27) and obtain the complete set of S, P and D phaseshifts. The knowledge of the D phaseshifts following from the correlation relation (30) allows to give the co-ordinates $u'_+(k)$ (see Eq. (27*a*)) and $v'_+(k)$ (see Eq. (27*b*)) of a « trial » point P'_1 (see Fig. 3), which moves along a certain auxiliary track in the (UV) plane. The correct position on this auxiliary track could then be fixed applying the method already used in the S and P wave approximation. Because of the experience gained in the analysis performed with S and P waves only, we feel that the phaseshift analysis of experimental data will be easier and quicker resorting to the mechanical or graphical translation of this parametrization procedure, based on the following equation

$$u'_+ + 2\cos 2\Delta\beta = 6 - 2\left(A_0 + \frac{A_2}{3} + \frac{A_4}{9}\right), \tag{33}$$

where $\Delta\beta = \beta_{35} - \beta_{33}$. With reference to Fig. 3, Eq. (33) simply states that the vector of lenghth 2 having origin in P'_1 and end-point lying at the distance from the V_+-axis, given by its right hand-side, has to form with the U_+-axis an angle $2\Delta\beta$, which is just the angle of the D-linkage in D_2 (equal to $2\beta^{(0)}_{33}$ for $A_4 = 0$). In this way the problem is reduced to a normal S and P wave approximation problem, with the only difference that the parameter w'_+ of the frozing condition, equivalent to (8*c*), depends now on both S and D phaseshifts. In fact, considering the sum $|\boldsymbol{a}_0|^2 + |\boldsymbol{b}_1|^2 + |\boldsymbol{b}_2|^2$ the following relation is readily obtained

$$\cos 2\alpha_3 + \cos 2(\alpha_{33} - \alpha_{31}) = 3 - \cos 2(\beta_{35} - \beta_{33}) - 2\left(A_0 - \frac{A_1}{9}\right) - \tfrac{1}{2}\,\mathrm{Re}\,(\boldsymbol{a}_0\boldsymbol{a}_2^*) \tag{34}$$

where, according to (24),

$$\mathrm{Re}\,(\boldsymbol{a}_0\boldsymbol{a}_2^*) = 3\cos 2(\alpha_3 - \beta_{35}) + 2\cos 2(\alpha_3 - \beta_{33}) - 5\cos 2\alpha_3 - (U_+ - u'_+) + 5\,. \tag{35}$$

In spite of this complication, the determination of the set $(\alpha_3, \alpha_{31}, \alpha_{33})$ satisfying, together with the previously fixed D-phaseshifts, Eq. (27) and (34) is rather quick, and the operational procedure should be obvious.

A complete discussion of this method, including Coulomb effects, will be given in a paper on phaseshift analysis of single channel reactions, in connection with the scattering of nucleons by zero-spin nuclei.

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SUPPLEMENTO AL VOLUME III, SERIE X
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Oscillations and Fluctuations in Gas Discharges.

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(ricevuto il 26 Marzo 1956)

CONTENTS. — **1.** Introduction. - **2.** Forms of disturbance. – **3.** Electron oscillations. – **4.** Ion oscillations.

1. – Introduction.

When a gaseous conductor is included in a circuit, it is frequently found that oscillations or irregular fluctuations occur. These may limit the range of useful application of devices used in connexion with nuclear and other investigations. The conductors showing these effects range from small glow-discharge tubes passing currents with density less than 1 μA/cm^2 to heavy current open and enclosed arcs, and it is almost certain that no form is entirely steady under all operating conditions. The present paper is an attempt to classify the forms of disturbance encountered, on the basis of experience with low pressure tubes in this laboratory, Brownian fluctuations of the electron and ion clouds and streams being generally excluded.

2. – Forms of Disturbance.

A rough distinction may be made between:

a) effects in which the gaseous conductor acts as a circuit unit possessing unusual properties, and

b) effects in which internal disturbances occur in the gaseous conductor which are practically independent of the external circuit in so far as it does not control the current through the gas.

Both may be affected by magnetic fields, and in ionospheric and astrophysical plasmas, by gravitational fields.

In class *a*) the gaseous conductor may be regarded to a first approximation as possessing differential resistance (R), capacity (C), and inductance (L), on account of its internal structure. The existence of negative values of R has long been known from the characteristic curves of arcs. A contribution to C can come from localized space-charges [1]. The existence of an L is less obvious. VERHAGEN [2] and VAN GEEL [3] have made a theoretical analysis of conditions in some simple ionized systems which shows how the gaseous conductor comes to possess properties which may be formally described by an R, C and L, which may be positive or negative, and have supported their conclusions by bridge and other measurements.

The significance of the quantities L and C can also be illustrated from the following idealized considerations. Let us suppose that the conductor has reversible energy U which depends on the terminal potential difference V and current i passing, through the relation

$$\delta U = \frac{\partial U}{\partial V}\,\delta V + \frac{\partial U}{\partial i}\,\delta i\,. \tag{1}$$

We ordinarily think of the capacity and self-inductance as associated with the energy U' of a system through the relation

$$U' = \tfrac{1}{2}CV^2 + \tfrac{1}{2}Li^2\,, \tag{2}$$

or the differential expression

$$\delta U' = CV\delta V + Li\delta i\,, \tag{3}$$

with C and L constants.

Comparing (1) and (3), we should have for the apparent capacity and inductance of the discharge the expressions

$$C = \frac{1}{V}\frac{\partial U}{\partial V}\,, \qquad L = \frac{1}{i}\frac{\partial U}{\partial i}\,. \tag{4}$$

There is no reason why C or L defined by (4) and measured by an appropriate circuit should not be either positive or negative. Further, if U is taken

[1] E. V. APPLETON and E. C. CHILDS: *Phil. Mag.*, **10**, 969 (1930).
[2] C. J. D. M. VERHAGEN: *Thesis* (Delft, 1942); *Physica*, **8**, 362 (1941).
[3] C. VAN GEEL: *Thesis* (Delft, 1955); *App. Sci. Res.*, B **5**, 79 (1955).

to be the electrostatic energy of the space-charges in the discharge, it may readily be shown that, for a simple discharge such as a glow with normal cathode fall, C and L defined by (4) may have the experimentally found orders of magnitude. With increasing complexity of structure in the discharge, the use of such simple procedure and concepts naturally becomes progressively less valid and they must be employed with caution. The effective impedance of a gaseous conductor also depends on frequency in a less simple way than that of a metallic conductor, because of time lag in changing the state of ionization ([3]), and of the possibility of resonances with disturbances of class *b*).

Internal disturbances *b*) are of great complexity, and still far from completely analyzed. The problems involved are to some extent like those which arise in the hydrodynamics of compressible fluids, complicated by the simultaneous presence of at least two interacting « fluids », electrons and positive ions, with both surface and volume sources and sinks. There is some evidence that phenomena akin to hydrodynamical turbulence can occur, particularly in presence of magnetic fields ([4,5]). There is also no doubt that many of the disturbances observed in practice are of too large amplitude for a linear theory of them to be valid.

In absence of magnetic fields, the linear theory of undamped longitudinal waves in plasmas, initiated by Tonks and Langmuir ([6]), and J. J. Thomson and G. P. Thomson ([7]), brings out the fundamental importance of two frequencies, the plasma-electron frequency ν_e and the plasma-ion frequency ν_p, with values

$$\nu_e = \{n_e e^2/\pi m\}^{\frac{1}{2}}, \qquad \nu_p = \{n_p e^2/\pi M\}^{\frac{1}{2}},$$

where n_e and n_p are the electron and ion concentrations, and m and M the mass of an electron and ion respectively. In the elementary theories, ν_e appears as a unique frequency of coherent electron vibration, and ν_p as an upper frequency of coherent ion vibration. For discharges in devices such as low pressure thyratrons and enclosed arcs, the numerical values of ν_e and ν_p are of the order of 10^3 MHz and 1 MHz respectively. It can scarcely be a complete coincidence that many of the oscillations generated by these devices fall into two bands, one narrow and close to ν_e ([8]), the other broad and extending from

([4]) A. Guthrie and R. K. Wakerling: *The Characteristics of Electrical Discharges in Magnetic Fields* (New York, 1949).

([5]) K. G. Emeleus: *Proc. Phys. Soc.*, **64** B, 166 (1951).

([6]) L. Tonks and I. Langmuir: *Phys. Rev.*, **33**, 195, 990 (1929).

([7]) J. J. Thomson and G. P. Thomson: *Conduction of Electricity Through Gases*, vol. II (Cambridge, 1933).

([8]) E. B. Armstrong and K. G. Emeleus: *Proc. Inst. Electr. Eng.*, **96** (III), 390 (1949).

about ν_p downward ([9,10]). The frequency ν_p can also however appear in other connexions, such as the oscillation of ions in a cathode potential minimum ([11]), and conditions generally in the gaseous conductors are certainly more complicated than was contemplated in the original theories (Sects. 3, 4). Later theoretical work summarized by GABOR ([12]) and ECKER ([13]) has been mainly concerned with electron oscillations. It indicates one source of large amplitude disturbances in the trapping of electrons in potential waves. The mathematical development of the fuller theories is still however incomplete and controversial ([14]).

When disturbances of class *b*) occur over a limited range of discharge conditions, it has been found in at least two instances (moving striations ([2]), oscillating anode spots ([15])) that strong coupling can be effected with the remainder of the circuit near the limiting conditions for which the class *b*) oscillations disappear.

3. – Electron Oscillations.

With these, as with the ion oscillations, it is necessary to enquire how they are excited and detected. There can be little doubt that most or all plasma-electron oscillations which have been studied hitherto are excited by primary approximately monoenergetic electrons with speed higher than that of the plasma electrons. The primary electrons may come from the hot cathode maintaining the discharge or from an auxiliary cathode ([16]). The primary beam itself oscillates, and so provides a ready means of detecting the presence of the oscillations. This general picture was first given clearly by MERRILL and WEBB, with the primary beam, that from the discharge cathode ([17]). Their experiments, repeatedly confirmed in this respect in this laboratory ([18,19]), show however that the oscillations increase very slowly in the first few mm outside the cathode sheath, and thereafter build up very rapidly in a mm or less. The visible structure and distribution of oscillation intensity nearer

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the anode is complex. MERRILL and WEBB described a series of layers, parallel to the first oscillating sheet. We have not been able to reproduce the pattern they describe, although a single visible first sheet has always been present in our tubes, in the shape of a meniscus convex to the cathode surface, if the latter is plane. To the anode side of this the primary electrons can frequently be seen to form both converging and diverging cones, indicating the presence of strong lateral fields in the meniscus, as well as the longitudinal fields of the elementary theory. Both probe and spectroscopic experiments ([20]) show that the primary electrons acquire a spread of energy in traversing the meniscus; it is not uncommon to find, for example, electrons with 35 eV energy to the anode side of the meniscus when the maximum energy to the cathode side is not greater than 25 eV and there is no detectable change in mean space-potential through the meniscus. A laterally restricted primary beam is found to be considerably broadened after passing where the meniscus would have been if a beam of greater cross-section had been used.

Qualitatively, the observations may be explained by supposing that a longitudinal slipping stream oscillation starts between the primary and plasma electrons immediately outside the cathode sheath, growing catastrophically at the meniscus, and having its principal frequency maintained close to the plasma-electron frequency. The longitudinal disturbances are accompanied by lateral disturbances because of radial inhomogeneity of the system. At the meniscus, whose presence may mark the attainment of a critical density of trapped electrons (Sect. 2), the state of the gas may be described by saying that it is the seat of a local self-maintained u.h.f. discharge, of which the amplitude is limited by factors which include loss of energy in inelastic collisions between the electrons and the gas. It seems impossible at the present time to give a more accurate quantitative account of what occurs.

When the cathode fall is increased from small values, for which the above description applies, to several hundred volts, or when the primary beam is reflected from a probe, effects similar to those obtained with a reflex klystron are obtained, and there is evidence that standing waves can be produced ([16,21]). The exact nature of the latter is not yet clear, and the klystron analogy must certainly not be over-stressed.

4. – Ion Oscillations.

Although disturbances with frequency components less than a few MHz are usually referred to as ion waves because they are in the correct band (Sect. 2), it has been questioned if true plasma ion waves of *small* amplitude

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have ever been definitely observed. The familiar positive moving striations in long commercial tubes are large amplitude disturbances; the faster negative waves which may accompany them ([22]) move through the large inhomogeneities produced by the positive waves in the plasma. The « noise » which originates near hot cathodes in low pressure tubes is also associated with disturbances having a continuous spectrum extending up to at least 100 MHz ([23]). It is not yet known if the latter are related to some sheath oscillations recently reported by GABOR *et al.* ([24]) in the same megacycle frequency band. The « noise » in the regular ion-frequency band (Sect. 2) is thought by us to arise from:

a) oscillations of ions in the potential minimum near a hot cathode emitting a space-charged limited current, possibly with associated electron oscillations in a shallow potential maximum ([25]);

b) irregular movement of the boundary between parts of the cathode emitting space-charge limited and temperature-limited currents;

c) a flicker effect at the cathode, possibly exaggerated by presence of ions, and possibly also associated with mode-jumping of standing space-charge waves in the gas very close to the cathode ([26]).

The « noise » is propagated with high speed (probably $> 10^7$ cm/s) ([27]) which is compatible with it being carried from the neighbourhood of the cathode by modulated primary beams.

The second and third factors (*b*, *c*) suspected to cause the « noise » are particular cases of a more general form of discharge instability, in which the internal structure of the gaseous conductor jumps abruptly from one almost stable form to another, through a continuous series of rapidly traversed unstable intermediate states.

It seems to be fairly generally agreed that most of the disturbances referred to in this section originate at or close to either anode or cathode, or where there is some irregularity, such as a change in shape or area of the cross-section of the discharge tube. It may be conjectured, however, that another, entirely different class of ion oscillations can occur, which are a close analogue for fast ions of the electron beam oscillations mentioned in Sects. 2 and 3. These may contribute to the instability of anode glows in low-current, low voltage tubes, but for their full development would require ion beams of much higher energy than exist near the anode in glow-discharges or ordinary arcs.

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SUPPLEMENTO AL VOLUME III, SERIE X
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Phaseshift Analysis of Pion-Nucleon Scattering Experiments. II. Operational Procedure.

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Introduction.

In a previous paper [1] some features concerning the phaseshift analysis of pion-nucleon scattering experiments have been discussed in the light of the kinematical properties of a linkage system, which visualizes with its mouvement the phaseshift evolution with increasing pion energy. It has been already shown how the mathematics of the phaseshift equations rule the behavior of the analyzer along the causal P-track for isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$, and how the phaseshifts must be read in order to fulfil their continuity requirements. In the previous discussion, however, the angular distribution coef-

[1] E. CLEMENTEL and C. VILLI: *Supplemento Nuovo Cimento* **3**, 474 (1956). This paper will be quoted as I.

ficients A_i, B_i, C_i and the analyzer input parameters $u_i(k)$, $v_i(k)$, $w_i(k)$ have been implicitely considered as perfectly defined numbers. In this paper we shall take into account the experimental uncertainties on the angular distribution coefficients and their influence on the analyzer input parameters, with the aim to outline an operational procedure suitable to deal with practical cases.

The limitations on the input parameters imposed by the mathematical requirements of the problem are discussed in Sect. **1**, and the criterion for testing the phaseshift stability for isotopic spin state $\frac{3}{2}$ is given in Sect. **2**. A general discussion on the determination of the phaseshifts for isotopic spin $\frac{1}{2}$ and for low energy positive scattering is carried out in Sect. **3**. A pathological situation, arising for isotopic spin $\frac{1}{2}$ at the scattering energy of 150 and 170 MeV, is pointed out in Sect. **4**. The procedure is illustrated in detail for the scattering energy of 189 MeV in Sect. **5**. The Appendix contains Tables and diagrams of general validity, which are useful to check the consistency of any solution either for isotopic spin $\frac{1}{2}$ up to about 200 MeV or for positive scattering at energies lower than about 70 MeV.

It is hoped that this investigation, which obviously does not intend to compete with electronic computers for hunting for the best solution at a given scattering energy, will somehow contribute to a better understanding of the results obtained with them.

Symbols and Notation.

$T = i/2$ $(i = 1, 3)$	total isotopic spin.
$A_+ = A_3$, $B_+ = B_3$, $C_+ = C_3$	angular distribution coefficients for positive scattering in S and P waves (Eq. (I.3)).
A_-, B_-, C_-; A_0, B_0, C_0	angular distribution coefficients for negative (direct and charge-exchange) scattering in S and P waves.
A_1, B_1, C_1	auxiliary coefficients for the determination of the phaseshifts for isotopic spin state $\frac{1}{2}$ (Eq. (I.5)).
$u_i(k)$, $v_i(k)$, $w_i(k)$	analyzer input parameters for the scattering in $T = i/2$ state (Eq. (I.9)).
α_i	S wave phaseshift for $T = i/2$ state.
$\alpha_{i3}^{(\pm)}$, $\alpha_{i1}^{(\pm)}$	Yang (+) and Fermi (—) P-phaseshifts for $T = i/2$ state (Eq. (I.19), Eq. (I.22)).
$\sigma_+(\vartheta) = \sigma_3(\vartheta)$, $\sigma_-(\vartheta)$, $\sigma_0(\vartheta)$	differential cross-section for the process $\pi^+ \rightarrow \pi^+$, $\pi^- \rightarrow \pi^-$ respectively $\pi^- \rightarrow \pi^0$.

$\sigma_+(k) \equiv \sigma_3(k)$, $\sigma_-(k)$, $\sigma_0(k)$ — total cross-section for the process $\pi^+ \to \pi^+$, $\pi^- \to \pi^-$ respectively $\pi^- \to \pi^0$.

$\sigma_1(\vartheta)$, $\sigma_1(k)$ — differential, total auxiliary cross-section (Eq. (I.6)).

$M = \sum_{x=A,B,C} \left\{ \frac{x(\text{exper}) - x(\text{theor})}{\Delta x} \right\}^2$ where $M \equiv M_\pm$ for $y = A_\pm$, $B_\pm$, $C_\pm$ and $M \equiv M_0$ for $y = A_0$, B_0, C_0.

$\mathcal{M} = M_+ + M_- + M_0$

$N = \sum \left\{ \frac{\sigma(\text{exper}) - \sigma(\text{theor})}{\Delta\sigma} \right\}^2$ where $N \equiv N_\pm$ for $\sigma \equiv \sigma_\pm(\vartheta)$ and $N \equiv N_0$ for $\sigma \equiv \sigma_0(\vartheta)$.

$\mathcal{N} = N_+ + N_- + N_0$

1. – The Ellipsoid of Errors. The Phaseshift Compatibility Region.

Let us indicate with $A_i = A_i^{(0)} \pm \Delta A_i$, $B_i = B_i^{(0)} \pm \Delta B_i$, $C_i^{(0)} \pm \Delta C_i$ the experimental angular distribution coefficients. For $i = 3$ they are obtained from the least squares fit of Eq. (I.1) on the differential cross-section for positive scattering. For $i = 1$ they are evaluated from Eqs. (I.5) using all nine experimental coefficients concerned with the processes $\pi^+ \to \pi^+$, $\pi^- \to \pi^-$, $\pi^- \to \pi^0$. In this case ΔA_1, ΔB_1, ΔC_1 are intended to be root-mean-square errors. For convenience of later discussion we shall replace C_i with $S_i = S_i^{(0)} \pm \Delta S_i$. Using Eqs. (I.9) the analyzer input parameters are easily obtained: $u_i(k) = u_i^{(0)}(k) \pm \Delta u_i$, $v_i(k) = v_i^{(0)}(k) \pm \Delta v_i$, $w_i(k) = w_i^{(0)}(k) \pm \Delta w_i$, where $u_i^{(0)}(k)$, $v_i^{(0)}(k)$, $w_i^{(0)}(k)$ are calculated from the central values of the primary data $A_i^{(0)}$, $B_i^{(0)}$, $S_i^{(0)}$ and the root-mean-square errors Δu_i, Δv_i, Δw_i are evaluated in terms of the experimental uncertainties ΔA_i, ΔB_i, ΔS_i.

We introduce the space (A, B, S), related to the cross-sections and the space (u, v, w), required by the mechanical analysis. The relation

$$M_i = \sum_{\substack{X_i = A_i, B_i, S_i \\ \text{or } X_i = u_i, v_i, w_i}} \left\{ \frac{X_i - X_i^{(0)}}{\Delta X_i} \right\}^2, \tag{1}$$

defines the ellipsoid of errors in the (A, B, S)-space or in the (u, v, w)-space. The ellipsoid center is given by the central values of the primary data respectively by the central values of the input parameters; the ellipsoid semiaxes are determined by the experimental uncertainties respectively by the root-mean-square errors on the input parameters. From the non-linear relation existing between $v_i(k)$ and the angular distribution coefficients (Eq. (I.9*b*)), it follows that the ellipsoid defined in one of these two spaces is deformed when

converted to the other. An example of this deformation is given in Sect. 5 for positive scattering at 189 MeV.

The exploration of the ellipsoid surface is required for testing the phaseshift stability and calculate their errors. In Sect. 3 it will be shown that the ellipsoid surface can be explored only for positive scattering ($i = 3$) at energies higher than ~ 70 MeV.

An important feature of the problem follows from the correlation existing between the three input parameters $u_i(k)$, $v_i(k)$, $w_i(k)$, since the choice of two of them limits the variability of the third one. This circumstance, although obvious it may be, cannot be simply stated on mathematical grounds. For this reason the mechanical system has been used to search the upper and lower limits of variability of $w_i(k)$ once $u_i(k)$, $v_i(k)$, i.e. the position of the terminal P_1 (I, Fig. 1), is arbitrarily fixed. The correlation function $w_i(k) = F[u_i(k), v_i(k)]$, determined in this way, describes in the (u, v, w)-space the boundary of a certain region: if solutions of the phaseshift equations exist, they must te found within this region. We shall call it the phaseshift compatibility region. The mechanical solution of the equation $w_i(k) = F[u_i(k), v_i(k)]$ is given in Appendix B. In Appendix A we have listed the co-ordinates of some tracks chosen within the compatibility region and the associated sets of phaseshifts. These values are of general validity because obtained regardless of any particular scattering situation. They have been determined taking advantage only of the kinematical properties of the analyzer, which are implied by the S and P wave approximation and by the assumption of the charge independence of nuclear forces. A three dimensional view of the ini-

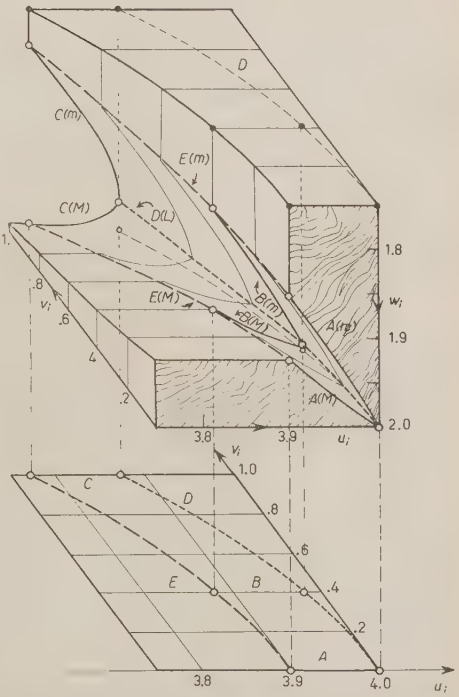

Fig. 1. – Three dimensional view of the compatibility region in proximity of the point $u_i=4$, $v_i=0$, $w_i=2$ and its projection on the (u, v)-plane (plane of the analyzer). The coordinates of some points on the explored tracks are listed in the corresponding Table of Appendix A and B.

tial part of the compatibility region is shown in Fig. 1, together with the explored tracks.

The equation $w_i(k) = F[u_i(k), v_i(k)]$ can easily be transformed into the (A, B, S)-space. As a result of this transformation (Eqs. (I.9)) one finds the new correlation function $A_i = \Phi(B_i, S_i)$, which describes the boundary of the compatibility region in the (A, B, S)-space (Appendix B). The important conclusion is that, once the values of two out of three primary data are fixed, the limits of variability of the third coefficient are established by the experimental uncertainties provided they do not fall outside the compatibility region. In fact, all points outside this region are consistent with the data, but conflicting with the mathematics of the problem. In Sect. 3 it will be shown that this circumstance is essential to understand the background of the phaseshift determination for isotopic spin $\frac{1}{2}$ and for low energy scattering in the isotopic spin $\frac{3}{2}$.

2. – Positive Scattering at Energies Higher than ~ 70 MeV.

To understand clearly the situation outlined in Sect. 1, let us consider separately the scattering of positive from that of negative pions. For $i = 3$ and for scattering energies higher than about 70 MeV, the compatibility region does not play an important role, because it is enough extended in the (A, B, S)-space, or in the (u, v, w)-space, to include within it almost entirely the ellipsoid of errors. In this case, the whole of the ellipsoid volume is therefore compatible both with the data and the mathematics of the problem. A strong correlation exists between the central values of the data (center of the ellipsoid) and the central values of the phaseshifts $\alpha_3^{(0)}$, $\alpha_{33}^{(0)}$, $\alpha_{31}^{(0)}$, as well as between the experimental uncertainties (semi-axes of the ellipsoid) and the uncertainties $\Delta\alpha_3$, $\Delta\alpha_{33}$, $\Delta\alpha_{31}$ on the phaseshifts. For this reason and from the point of view of the analysis of scattering experiments, the determination of the phaseshifts for isotopic spin state $\frac{3}{2}$ above ~ 70 MeV is a trivial matter.

The procedure to be followed is obvious. From the central values of the primary data we calculate the central values of the analyzer input parameters. Then, the central values of the phaseshifts $\alpha_3^{(0)}$, $\alpha_{33}^{(0)}$, $\alpha_{31}^{(0)}$ are measured at the center of the ellipsoid of errors $u_i^{(0)}(k)$, $r_i^{(0)}(k)$, $w_i^{(0)}(k)$. To determine the errors $\Delta\alpha_3$, $\Delta\alpha_{33}$, $\Delta\alpha_{31}$ on the phaseshifts we shall explore the ellipsoid surface, choosing on it 26 points, according to a symmetry criterion with respect to the ellipsoid center. The coordinates of these points ($M_3 = 1$) are listed in Table I. The linkage system produces quickly 26 sets of phaseshifts $\alpha_3^{(n)}$, $\alpha_{33}^{(n)}$, $\alpha_{31}^{(n)}$, associated to the $n = 1, 2, 3 \ldots 26$ points. These sets are then used in the following relation to get the final answer

$$\alpha = \alpha^{(0)} \pm \left\{ (1/25) \sum_{n=1}^{n=26} [\alpha^{(n)} - \alpha^{(0)}] \right\}^{\frac{1}{2}} = \alpha^{(0)} \pm \Delta\alpha \,, \tag{2}$$

TABLE I. – *Co-ordinates of 23 points on the ellipsoid surface for testing the phaseshift stability against variations of the angular distribution coefficients or against variations of the analyzer input parameters.*

No. of points	Coordinates		
	$X = A_3$ or $u_3(k)$	$Y = B_3$ or $v_3(k)$	$Z = S_3$ or $w_3(k)$
2	$\pm \Delta X$	0	0
2	0	$\pm \Delta Y$	0
2	0	0	$\pm \Delta Z$
4	$\pm (2)^{-\frac{1}{2}} \Delta X$	$\pm (2)^{-\frac{1}{2}} \Delta Y$	0
4	$\pm (2)^{-\frac{1}{2}} \Delta X$	0	$\pm (2)^{-\frac{1}{2}} \Delta Z$
4	0	$\pm (2)^{-\frac{1}{2}} \Delta Y$	$\pm (2)^{-\frac{1}{2}} \Delta Z$
8	$\pm (3)^{-\frac{1}{2}} \Delta X$	$\pm (3)^{-\frac{1}{2}} \Delta Y$	$\pm (3)^{-\frac{1}{2}} \Delta Z$

where $\alpha = \alpha_3, \alpha_{33}, \alpha_{31}$. In this case $\mathcal{M}$ equals the « accuracy » of the least squares fit on the differential cross-section and $M_+ = 0$. In practical cases M_+ will be found slightly different from zero because of errors in reading the phaseshifts on the analyzer. The condition which must be satisfied is that the actual value of M_+ be negligible compared to 1. It is believed to be rather academical to improve the phaseshift analysis beyond the tolerable value $M_+ \sim 0.005$.

3. – Low Energy Positive Scattering (< 70 MeV) and Scattering in the Isotopic Spin $\frac{1}{2}$ up to ~ 200 MeV.

An entirely different situation arises when the linkage system works at the extreme limit of the (u, v)-plane (I, Fig. 1; Fig. 1). As it has already been pointed out (I, Sect. 3), this region belongs either to low energy positive scattering (< 70 MeV) or to the scattering in the isotopic spin state $\frac{1}{2}$ up to about 200 MeV. It is clear that in S and P wave approximation, the portion of the ellipsoid in the (u, v, w)-space, the coordinates of which do not satisfy the conditions $u_1(k)\, \Psi \leqslant 4$ $(\sigma_1(k) \geqslant 0)$, $u_1^2(k) + v_1^2(k) \leqslant 16$, $v_1(k)$ real, $w_1(k) \leqslant 2$ $(A_1 \geqslant 0)$ must be cut-off, as mathematically inconsistent. All the points which are left in the remaining portion of the ellipsoid are of course compatible with the original data, but not all of them are compatible with the phaseshift equations either, since the compatibility region happens to be only a fraction of the available ellipsoid volume. The original ellipsoid of errors is therefore destroyed and the criterion used in Sect. 2 to determine the phaseshifts and to test their stability becomes meaningless in this case. To be definite, in the following we shall refer to the scattering in the isotopic spin state $\frac{1}{2}$ only.

Two limiting cases must be taken into account: (I) the central values of the input parameters $u_i(k)$, $v_i(k)$, $w(k)_i$ fall within the compatibility region, but the ellipsoid of errors is partially outside; (II) the center of the ellipsoid of errors is outside the compatibility region.

In the former case it is still possible to measure the central values of the phaseshifts, but a new criterion is required to evaluate their uncertainty. Such a criterion could be to explore the available region in the (u, v, w)-space and then apply equation (2). Apart from the arbitrariness in the choice of the points of the compatibility region, within the ellipsoid of errors, it is clear that when case (I) occurs only a weak correlation exists between the phaseshift variation and the variation of the data. In fact, the former are bound to occur within the compatibility region, whereas the latter may occur also outside, as it actually happens. It follows that in this case any limited variation of the phaseshifts, revealed by the analysis, is mainly a consequence of the limitations imposed on them by the existence of the compatibility region, which sharply cuts off the experimental interval of variation. Unfortunately, according to the available experimental data, it seems that case (I), which at least allows the determination of the central values of the phaseshifts, is less frequent than case (II), for which the situation appears far more confused.

It is clear that when case (II) occurs, a criterion must be devised to determine the « central » values of the phaseshifts. We shall not discuss the problem concerning the determination of the errors of the phaseshifts, because we believe it to be rather objectionable in this case.

The following three situations belong to case (II): *a*) the ellipsoid center does not exist in real (u, v, w)-space ($r_1(k)$ imaginary); *b*) the ellipsoid center exists in real (u, v, w)-space but outside the « kinematical » region, i.e. the region available to the linkage system (either $\sigma_1(k) < 0$, i.e. $u_1(k) > 4$, or $A_1 < 0$, i.e. $w_1(k) > 2$); *c*) the ellipsoid center falls within the « kinematical » region.

The very problem concerned with case (II) is to replace the original center of the ellipsoid with a properly chosen point on the boundary of the compatibility region, and then hunt for the best solution around that point. To unify the procedure for all the three possibilities previously considered, it is convenient to work in the (A, B, S)-space. The criterion we shall follow in order to readjust the data is the following: the original ellipsoid center $A_1^{(0)}$, $B_1^{(0)}$, $S_1^{(0)}$ will be replaced by a point A_1', B_1', S_1', chosen on the boundary of the compatibility region according to the condition that its distance from the original center be a minimum (in units of standard errors). With this new point we calculate the input parameters $u_1'(k)$, $v_1'(k)$, $w_1'(k)$ and derive the corresponding phaseshifts. Although no general rule can be given for the search of the best solution close to the point $u_1'(k)$, $v_1'(k)$, $w_1'(k)$, in several cases it has been found that the sets which make $\mathcal{H}$ minimum are obtained by a slight variation of $w_1'(k)$, keeping $u_1'(k)$ and $v_1'(k)$ fixed.

The minimum distance condition has been preferred with the obvious intent to alter as less as possible the original data.

Since no simple mathematical expression exists for the boundary surface of the compatibility region $A_i = \Phi(B_i, S_i)$ in (A, B, S)-space, as well as in (u, v, w)-space, the point A_1', B_1', S_1' cannot be analytically determined by searching the minimum of the relation

$$M_1 = \left(\frac{A_1' - A_1^{(0)}}{\Delta A_1}\right)^2 + \left(\frac{B_1' - B_1^{(0)}}{\Delta B_1}\right)^2 + \left(\frac{S_1' - S_1^{(0)}}{\Delta S_1}\right)^2 , \tag{3}$$

under the condition that the point A_1', B_1', S_1' lies on the boundary surface. For this reason the only possibility to determine the point A_1', B_1', S_1' is to resort to a graphical representation of the projections of the compatibility region on the planes (A_1B_1), (A_1S_1), (B_1S_1), using the coordinates of some limiting tracks listed in Appendix B. In this way it is fairly easy to enter the compatibility region along the minimum distance, starting from the original center A_1, B_1, S_1.

Since the probability to reproduce the original values of the auxiliary coefficients in terms of the phaseshifts is given by $\exp[-M_1/2]$, we assume that $M_1 < 1$ provides an indication that only accidental errors come into play. If confidence is given to both S and P wave approximation and charge independence hypothesis, a value $M_1 > 4$ may be retained as an indication that the experimental data are affected by systematic errors.

We observe that the two situations *a*) and *b*) can be reduced to situation *c*) by minimizing expression (3) under the following conditions

$$S_1^* \geqslant 0, \quad A_1^* \geqslant 0, \quad 3S_1^* + S_1^{*2} - 2A_1^* + B_1^* \geqslant 0, \quad (v_1^*(k) \text{ real}), \tag{4}$$

which move the original center $A_1^{(0)}$, $B_1^{(0)}$, $S_1^{(0)}$ into a new point A_1^*, B_1^*, S_1^*, the corresponding of which in the (u, v, w)-space lies in the «kinematical» region. On calling Λ the Lagrange multiplier, it is found

$$A_1^* = A_1^{(0)} - \Lambda(\Delta A_1)^2, \qquad B_1^* = B_1^{(0)} + (\Lambda/2)(\Delta B_1)^2 \tag{5}$$

where Λ is determined together with S_1^* by solving graphically the following two equations

$$\Lambda = \frac{2(S_1^{*2} - 3S_1^* + 2A_1^{(0)} - B_1^{(0)})}{4(\Delta A_1)^2 + (\Delta B_1)^2} , \tag{6a}$$

$$\Lambda = \frac{S_1^{(0)} - S_1^*}{(\Delta S_1)^2(S_1^* - \frac{3}{2})} . \tag{6b}$$

It is clear that the conditions (4) are necessary but not sufficient for the point A_1^*, B_1^*, S_1^* to lie on the boundary of the compatibility region and therefore the point will in general fall outside this region. A further graphical step, as illustrated above, is required to move along the minimum distance towards the point A_1', B_1', S_1', previously defined.

The final check for the best solution requires the evaluation of $\mathcal{M}$ and $\mathcal{N}$, both of them depending also on the phaseshifts for isotopic spin $\frac{3}{2}$. The phaseshifts α_3, α_{31}, α_{33} can be determined using the experimental coefficients for positive scattering (Sect. 2). However, another alternative has to be considered mainly when case (II) occurs, i.e. to take into account that the shift of the original ellipsoid center implies a combined variation of all the nine primary data concerned with the processes $\pi^+ \to \pi^+$, $\pi^- \to \pi^-$, $\pi^- \to \pi^0$ (Eqs. (I.5)). The new nine coefficients are readily obtained from their original central values by distributing the displacements $A_1^{(0)} - A_1'$, $B_1^{(0)} - B_1'$, $S_1^{(0)} - S_1'$ of the auxiliary coefficients among the latter according to the relations

$$X_3' = X_3^{(0)} - \frac{2(Y_1' - Y_1^{(0)})(\Delta X_3)^2}{(\Delta X_3)^2 + 9(\Delta X_-)^2 + 9(\Delta X_0)^2}, \tag{7a}$$

$$X_{\underline{0}}' = X_{\underline{0}}^{(0)} + \frac{6(Y_1' - Y_1^{(0)})(\Delta X_{\underline{0}})^2}{(\Delta X_3)^2 + 9(\Delta X_-)^2 + 9(\Delta X_0)^2}, \tag{7b}$$

where $X = A, B, S$ and $Y_1 = A_1, B_1, S_1$. With the new coefficients A_3', B_3', S_3' we evaluate the new input parameters $u_3'(k)$, $v_3'(k)$, $w_3'(k)$ and measure the new set of phaseshifts α_3', α_{31}', α_{33}'. The choice between the two sets, for a given $T = \frac{1}{2}$ solution, is made according to the minimum value of $\mathcal{N}$.

4. – Data-Phaseshifts Correlation for the $T = \frac{1}{2}$ State.

It is apparent that in situations classified as case (II) the correlation, if any, between the experimental data and the phaseshifts is certainly very weak. In fact, one should expect that ellipsoids having different centers in the (A, B, S)-space and belonging to different scattering energies may lead to the same set of phaseshifts. This circumstance arises of course because in case (II) the results of any analysis are far more strongly bound to the compatibility region than to the original data. It follows that when such a situation occurs, the one-to-one correspondence between data and phaseshifts is destroyed, i.e. for a given ellispoid of errors it is possible to establish a criterion for the determination of the phaseshifts, but starting from the phaseshifts, nothing can be said about the ellipsoid of errors they belong to. The following example will clear the situation.

TABLE II. – *Experimental coefficients A_i, B_i, S_i and input parameters $u_i(k)$, $v_i(k)$, $w_i(k)$ at 150 and 170 MeV.*

150 MeV			
i	A_i	B_i	S_i
1	-0.007 ± 0.020	-0.017 ± 0.022	-0.005 ± 0.026
3	0.773 ± 0.026	-0.237 ± 0.037	1.327 ± 0.035
i	u_i	v_i	w_i
1	4.011 ± 0.051	$0.279\sqrt{-1}$	2.014 ± 0.040
3	1.346 ± 0.070	1.322 ± 0.099	0.454 ± 0.052

170 MeV			
i	A_i	B_i	S_i
1	-0.037 ± 0.024	-0.004 ± 0.028	-0.044 ± 0.034
3	1.035 ± 0.089	-0.137 ± 0.052	1.830 ± 0.051
i	u_i	v_i	w_i
1	4.088 ± 0.067	$0.514\sqrt{-1}$	2.074 ± 0.048
3	0.340 ± 0.102	$0.510\sqrt{-1}$	-0.070 ± 0.078

Let us consider the scattering data at 150 and 170 MeV, recently obtained at Carnegie [2] (Table II).

The projection on the plane $B_1 = 0$ of the initial part of the compatibility region is shown in Fig. 2 (shadowed region).

The coordinates of the tracks projected on this plane have been evaluated using the Tables given in Appendix B. From Fig. 2 it is seen that, according to the experimental data of Table II, the ellipses, projections on the plane $B_1 = 0$ of the ellipsoids of errors, corresponding to $M_1 = 1$, are partially outside (150 MeV) or totally outside (170 MeV) the compatibility region. The same situation is also found by considering the projections of the compatibility region and of the ellipsoids of errors on the planes $A_1 = 0$ and $S_1 = 0$. The phaseshifts fitting the data at 150 MeV will be found only in that portion of the error ellipsoid contained in the compatibility region, where the original center has to be moved. To enter the compatibility region starting from the data at 170 MeV one has to readjust them outside the standard errors and

[2] J. ASHKIN, J. P. BLASER, F. FEINER and M. O. STERN: *Phys. Rev.*, **101**, 1149 (1956).

enlarge (arbitrarily) the ellipsoid. In particular, for $M_1 = 6$ the error ellipsoid fills up the same portion of the compatibility region as the ellipsoid having $M_1 = 1$ at 150 MeV. Therefore the phaseshifts must be practically the same, because determined by the compatibility region and largely independent from the original data. It has been found (2), in fact, $\alpha_1 = 9°$, $\alpha_{13} = -2°$, $\alpha_{11} = 2°$ at 150 MeV and $\alpha_1 = 10°$, $\alpha_{13} = -2°$, $\alpha_{11} = 0°$ at 170 MeV. However, these are not the only possible solutions. Using the analyzer it is readily found that the Carnegie solutions have the following associated solutions $\alpha_1 = 9°$, $\alpha_{13} = -0.5°$, $\alpha_{11} = 3.5°$ at 150 MeV and $\alpha_1 = 10°$, $\alpha_{13} = 0.6°$, $\alpha_{11} = -2.6°$, at 170 MeV, which satisfy the data equally well.

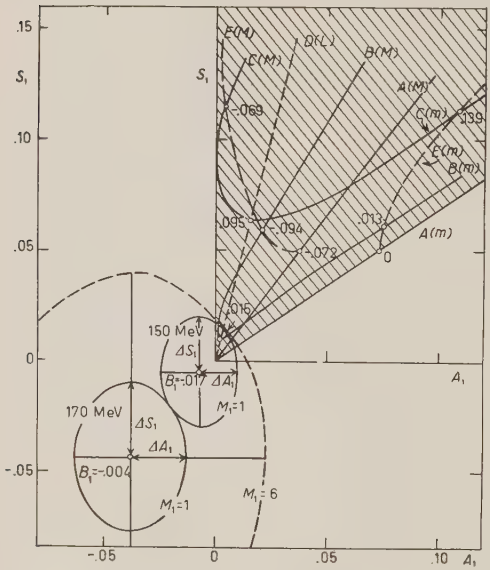

Fig. 2. – Projection on the plane $B_1 = 0$ of the initial part of the compatibility region and of the ellipsoid of errors for the auxiliary coefficients concerning the scattering of negative pions at 150 and 170 MeV. The coordinates of the projected tracks are given in Appendix B.

In conclusion, all the experimental data which fall or which are brought by a proper readjustement within the same portion of the compatibility region lead to the same set of phaseshifts. These phaseshifts are determined essentially by the mathematical requirements of the phaseshift equations and therefore can hardly reproduce the physical features of the scattering process. As a limiting example, let us consider the scattering of negative pions at 61.5 MeV, i.e. the scattering in an energy region far away from that of the Carnegie experiments. It is found, as a consequence of the readjustement of the data, that the portion of the compatibility region involved at this scattering energy is practically the same which belongs to the ellipsoid $M_1 = 1$ at 150 MeV and to the ellipsoid $M_1 = 6$ at 170 MeV. The phaseshifts must therefore be practically the same. In fact, at 61.5 MeV it has been found $\alpha_1 = 9°$, $\alpha_{13} = -2°$, $\alpha_{11} = 0.1°$ (I, Table I, solution I), which identifies practically with the Carnegie solution at 170 MeV. Therefore, once such a solution is found, nothing definite can be said about the ellipsoid of errors it belongs to.

All the manipulations which are required in order to enter the compatibility

region and the consequent readjustement of the primary data are automatically performed by electronic computers, which give of course mathematically correct solutions, without « pointing out » the correct significance to be attributed to them. The only clue that the background of the scattering situation is unreasonable is provided by the value of $\mathcal{T}$. This number is to a certain extent, at least for the scattering in the isotopic spin $\frac{1}{2}$, more significant than the phaseshifts.

5. – Analysis of Pion-Nucleon Scattering at 189 MeV.

To illustrate the method outlined in Sect. 3, we have chosen the scattering energy of 189 MeV. This energy is interesting both because for positive scattering it is close to the resonance energy and because it is higher than the inversion energy of the linkage system (I, Sect. 2). Use has been made of the experimental data of ANDERSON *et al.* [3]. The final results obtained with the linkage system are compared in Table III with the phaseshifts obtained by ANDERSON *et al.*, using an electronic computer.

TABLE III. – *Phaseshift analysis at* 189 MeV.

Phaseshifts	α_3	α_{33}	α_{31}
Present method	$-13.0^\circ \pm 4.7^\circ$	$97.5^\circ \pm 3.2^\circ$	$-10.4^\circ \pm 5.2^\circ$
ANDERSON *et al.*	$-11.3^\circ \pm 3.2^\circ$	$98.8^\circ \pm 3.6^\circ$	$-11.6^\circ \pm 5.1^\circ$

Phaseshifts	α_1	α_{13}	α_{11}	$\mathcal{T}$
Present method	17.5°	-1.6°	-1.6°	7.65
ANDERSON *et al.*	-2.8°	-2.1°	-2.6°	7.62

To evaluate $\mathcal{T}$ the summation has been taken over five angles for the process $\pi^+ \to \pi^+$ and over six angles both for the processes $\pi^- \to \pi^-$ and $\pi^- \to \pi^0$. In the latter case the Bodansky formula [4] has been used to take account of the counter efficiency $n(\gamma)$, given in Table X of ref. [3]. As far as the fit of the experimental data is concerned both sets of phaseshifts appear to be equally good. This can be seen in greater detail by inspection of Table V. However, it must be stressed that the Anderson set implies $r_1(k) < 0$, whereas out set

[3] H. L. ANDERSON, W. C. DAVIDON, M. GLICKSMAN and U. E. KRUSE: *Phys. Rev.*, **100**, 279 (1955).

[4] D. BODANSKY, A. M. SACHS and J. STEINBERGER: *Phys. Rev.*, **93**, 1367 (1954).

gives $r_1(k) > 0$. The former set seems therefore to be conflicting with the causality prescriptions of the scattering theory (I, Sect. 3). The striking difference between the two sets lies mainly in the value of α_1. The value $\alpha_1 = 17.5°$ is in agreement with the value $\alpha_1 = 15°$ recently obtained by OREAR (5) assuming $\alpha_{11} = \alpha_{13} = \alpha_{31} = 0$. Furthermore our α_1 joins with the Carnegie values of this phaseshifts at 150 and 170 MeV (2). The two Carnegie solutions (Sect. 4) are found in Table B of Appendix A for $u_1 = 3.930$, $v_1 = 0.400$ and w_1 somewhere between 1.928 and 1.910. However, we point out that a preliminary investigation, based on the minimum of M_1 and regardless the minimum of $\mathcal{M}$, which is of course essential for the final check, has shown that, according to the data listed in Table II, at 150 MeV solutions ought to be sought along the track $u_1 = 3.960$, $v_1 = 0.400$ and w_1 lying somewhere between 1.964 and 1.959 (Table B of Appendix A). This track depresses the value of α_1 to about $5° \div 7°$.

The least square coefficients of the differential cross-sections in S and P waves for elastic (positive and negative) and charge exchange scattering are listed in Table IV, together with the auxiliary coefficients, calculated by means of Eqs. (I.5), and the input parameters, calculated by means of Eqs. (I.9) for $i = 1$ and $i = 3$.

TABLE IV. – *Experimental coefficients* A_3, B_3, C_3, A_-, B_-, C_-, A_0, B_0, C_0, *auxiliary coefficients* A_1, B_1, C_1 *and input parameters* u_i, v_i, w_i *for* 189 MeV *pion-nucleon scattering.*

Experimental coefficients			Auxiliary coefficients and input parameters		
A_3	B_3	C_3	A_1	B_2	C_1
0.953 ± 0.100	0.408 ± 0.171	3.369 ± 0.344	0.056 ± 0.119	-0.217 ± 0.154	-0.066 ± 0.368
A_-	B_-	C_-	u_3	v_3	w_3
0.104 ± 0.018	0.034 ± 0.029	0.408 ± 0.055	0.152 ± 0.177	1.296 ± 0.640	0.094 ± 0.200
A_0	B_0	C_0	u_1	v_1	w_1
0.251 ± 0.070	0.043 ± 0.079	0.671 ± 0.210	3.932 ± 0.260	$+ 0.954\sqrt{-1}$	1.888 ± 0.238

The numbers listed in Table IV for $i = 3$ have been evaluated taking into account the energy spread of ~ 2 MeV, which implies on $k^2 = 0.1316$ (mb)$^{-1}$ a root-mean-square error equal to 0.0013 (mb)$^{-1}$. Now let us go into details.

(5) J. OREAR: *Phys Rev.*, **100**, 288 (1955).

5·1. *Scattering in the* $T = \frac{3}{2}$ *state.* – From the coefficients of Table IV it is found $S_3 = 2.076 \pm 0.089$, whereas the same quantity calculated from the experimental value of the total cross-section, obtained from transmission measurements, is 2.034 ± 0.054. To take account of the latter data we assume the weighted value $\bar{S}_3 = 2.045 \pm 0.072$. The coefficients A_3 and C_3 must therefore be readjusted. Since the angles at which the differential cross-section has been measured allow a better determination of A_3 than C_3 we keep fixed the value of A_3, given in Table IV, and calculate from $\bar{S}_3$ the new value $\bar{C}_3^{(0)} = 3.276$. The corresponding input parameters are $u_3 = -0.091 \pm 0.177$, $v_3 = -1.349 \pm 0.430$, $w_3 = 0.094 \pm 0.250$. The values in degrees of the Fermi phaseshifts α_3, $\alpha_{33}^{(-)}$, $\alpha_{31}^{(-)}$ at the center of the ellipsoid ($u_3^{(0)} = -0.091$, $v_3^{(0)} = -1.349$, $w_3^{(0)} = 0.094$) and the sets associated to the 26 points, given in Table I, are shown in Fig. 3.

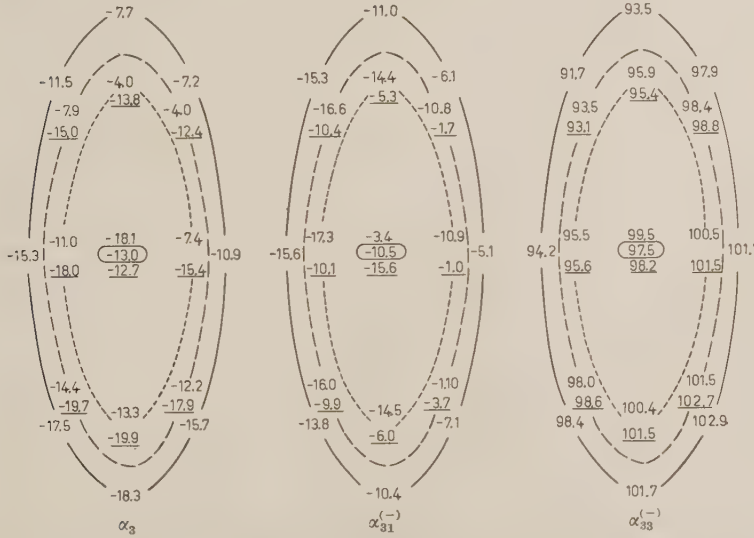

Fig. 3. – Values (in degrees) of the phaseshifts *a*) α_3, *b*) α_{31}, *c*) α_{33} at the center of the ellipsoid (central number at the center of the ellipses) and at the 26 points chosen on the ellipsoid surface. u_3 and v_3 are the horizontal respectively the vertical axis. The three ellipses for each phaseshift are level curves corresponding to $w_3 = w_3^{(0)}$ (full line), $w_3 = w_3^{(0)} \pm (3)^{-\frac{1}{2}} \Delta w_3$ (dashed line), $w_3 = w_3^{(0)} \pm (2)^{-\frac{1}{2}} \Delta w_3$ (dotted line). The underlined phaseshifts correspond to points lying on the lower half part of the ellipsoid surface.

The following set of Fermi phaseshifts have been found

$$(8) \quad \alpha_3 = \begin{matrix} -13.0^\circ \pm 4.7^\circ \\ (-12.9^\circ) \end{matrix}, \quad \alpha_{33}^{(-)} = \begin{matrix} 97.5^\circ \pm 3.2^\circ \\ (98.0^\circ) \end{matrix}, \quad \alpha_{31}^{(-)} = \begin{matrix} -10.4^\circ \pm 5.2^\circ \\ (-10.2^\circ) \end{matrix}.$$

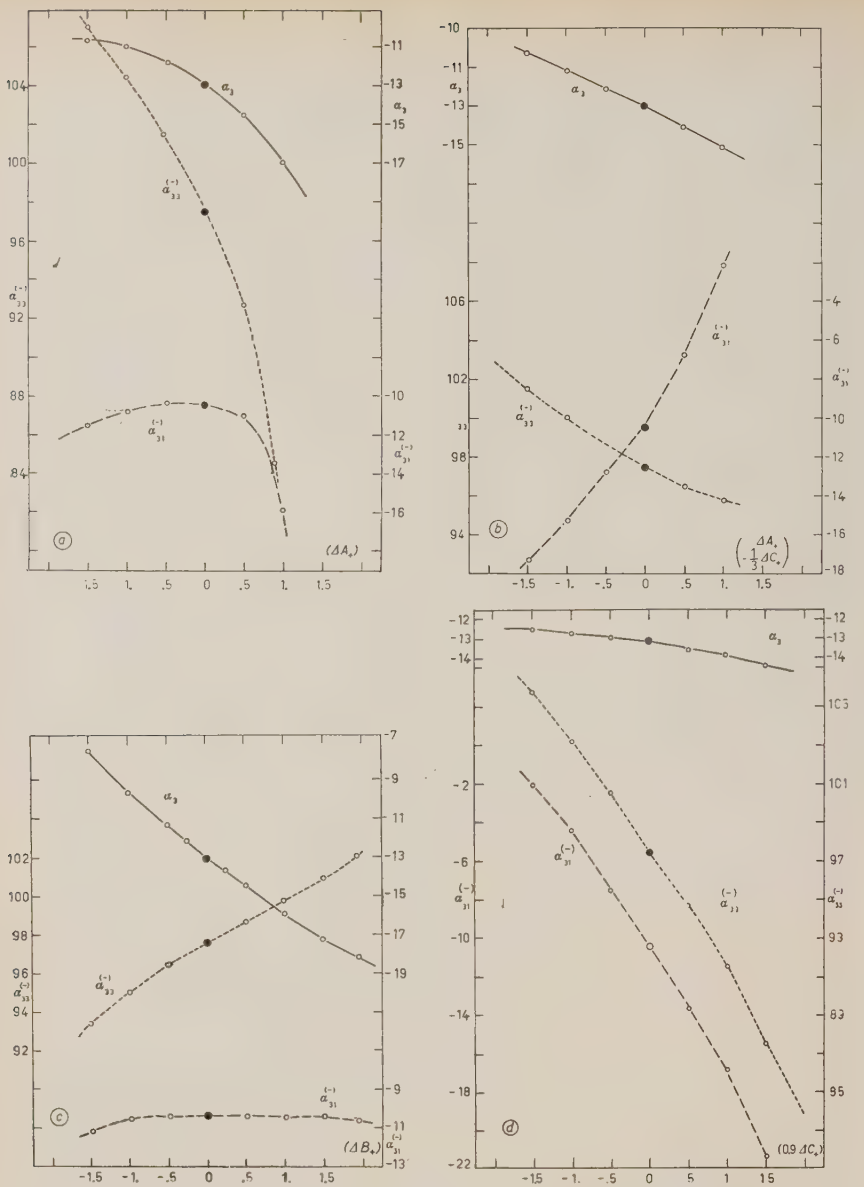

Fig. 4. – Behavior of α_3, $\alpha_{33}^{(-)}$, $\alpha_{31}^{(-)}$ against variations of the angular distribution coefficients from their central values $A_3^{(0)}=0.953$, $B_3^{(0)}=0.408$, $C_3^{(0)}=3.276$ ($\bar{S}_3^{(0)}=2.045$) up to $(3/2)\,\Delta A_3$, $(3/2)\,\Delta B_3$, $0.9(3/2)\,\Delta C_3$, $(\Delta\bar{S}_3)$. *a*) variation of A_3 (B_3 and C_3 are kept constant): $0.803\leqslant A_3\leqslant 1.103$; $(1.895\leqslant\bar{S}_3\leqslant 2.195)$; *b*) correlated variation of A_3 and C_3 (B_3 and $\bar{S}_3$ are kept constant): $0.803\leqslant A_3\leqslant 1.103$, $3.726\geqslant C_3\geqslant 2.826$; *c*) variation of B_3 (A_3, C_3 and $\bar{S}_3$ are kept constant): $0.253\leqslant B_3\leqslant 0.563$; *d*) variation of C_3 (A_3 and B_3 are kept constant); $3.051\leqslant C_3\leqslant 3.501$ $(1.970\leqslant\bar{S}_3\leqslant 2.120)$. The abscissa is given in error units: $\Delta A_3=0.100$, $\Delta B_3=0.170$, $0.9\cdot\Delta C_3=0.300$ $(\Delta\bar{S}_3=0.100)$.

The errors on the phaseshifts have been evaluated by exploring the ellipsoid surface (Sect. 2) and then using Eq. (2). The values into brackets are averages over the 26 points on the ellipsoid surface. It is seen that the calculated errors on the phaseshifts are very close to those calculated by ANDERSON *et al.*: $\Delta\alpha_3 = 3.2°$, $\Delta\alpha_{33} = 3.6°$, $\Delta\alpha_{31} = 5.1°$.

The phaseshift stability has been investigated by varying the analyzer configuration and measuring the corresponding variation of the phaseshifts (Fig. 4).

The four tracks, which have been explored, are shown in the (u, v)-plane in Fig. 5.

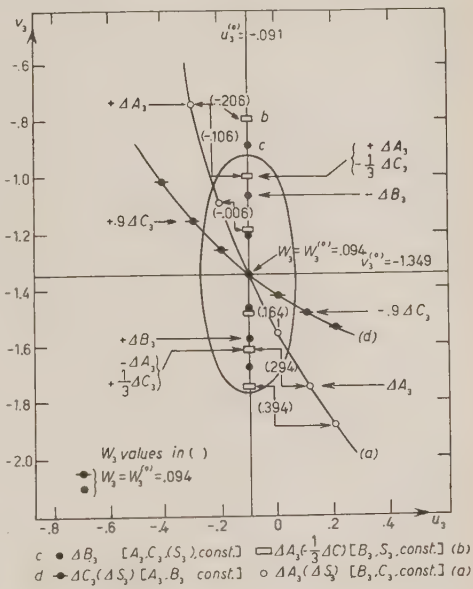

Fig. 5. – Projection on the plane $w_3^{(0)} = 0.094$ of the ellipsoid of errors for positive scattering at 189 MeV. The phaseshift variations, shown in Fig. 4, have been measured by moving the analyzer along the corresponding tracks *a*), *b*), *c*), *d*). The tracks *b*), *c*), *d*) correspond to the deformed axes A_3, B_3, S_3 of the ellipsoid of errors in (A, B, S)-space.

5·2. *Scattering in the* $T = \frac{1}{2}$ *state.* – The weighted value of $\overline{S}_3$ modifies slightly the value of $\overline{C}_1$, given in Table IV. It is found $C_1 = 0.019 \pm 0.368$. It follows that $S_1 = 0.050 \pm 0.156$. Inspection of the coefficients listed in Table IV shows that none of the inequalities (5) is satisfied within the whole interval of variation, allowed by quoted experimental errors. The center of the ellipsoid of errors $A_1^{(0)} = 0.056$, $B_1^{(0)} = -0.218$, $S_1^{(0)} = 0.050$ is outside the compatibility region, as it can be easily realized by inspection of the Tables given in Appendix B. The corresponding center in real (u, v, w)-space does not exist. It is clear that this scattering situation corresponds just to what has been classified as case (II, *a*) in Sect. 3.

Inspection of the diagrams obtained from Table A(m)-E(m) (Appendix B), shows that in order to approach, along the minimum distance, the compatibility region starting from the point $A_1^{(0)} = 0.056$, $B_1^{(0)} = -0.218$, $S_1^{(0)} = 0.050$, it is convenient to follow that track, the coordinates of which are listed in Table B(m). An alternative way to reach this track is to move the original

point $A_1^{(0)}$, $B_1^{(0)}$, $S_1^{(0)}$ within the «kinematical» region by minimizing Eq. (4) under the conditions (5). The calculation is straightforward and it is found $A_1^* = 0.035$, $B_1^* = -0.206$, $S_1^* = 0.095$ ($M_1 = 0.12$). The corresponding point in (u, v, w)-space is $u_1^* = 3.810$, $v_1^* = 0$, $w_1^* = 1.930$. From Tables A(m) and A(M) it is readily seen that this point is still outside the compatibility region. A further graphical step starting from the point A_1^*, B_1^*, S_1^* will lead again to the track B(M).

Several points have been considered along this track. Among them we have chosen the point lying a minimum distance apart (in units of standard errors) from the original center $A_1^{(0)}$, $B_1^{(0)}$, $S_1^{(0)}$. This point corresponds to $M_1 = 0.22$ and its coordinates are $A_1' = 0.055$, $B_1' = -0.185$, $S_1' = 0.116$. The analyzer input parameters, corresponding to these values, are $u_1' = 3.768$, $v_1' = \pm 0.400$, $w_1' = 1.890$. The phaseshifts given by the analyzer at this point are listed in Table V as solution I (non causal) ($\alpha_1 > 0$, $v_1' < 0$) and solution II (causal) ($\alpha_1 < 0$, $v_1' > 0$). In both cases the P-stage of the analyzer is characterized by the point Q_1 coinciding with the point Q_2 (I, Fig. 1) and no distinction can be made between solutions labelled with (+) and (—).

It is found that the value $w_1' = 1.890$ forbids solutions having α_1 and v_1' both positive or negative (Table B(M) and Table B). For this reason a slight penetration has been made within the compatibility region in order to find out, a minimum distance apart from the point u_1', v_1', w_1', a new point allowing solutions having α_1 and v_1 both positive or negative. The coordinates of this point are $u_1'' = 3.814$, $v_1'' = \pm 0.391$, $w_1'' = 1.819$ and the corresponding point in (A, B, S)-space is $A_1'' = 0.090$, $B_1'' = -0.052$, $S_1'' = 0.093$. This point lies a distance $(M_1)^{\frac{1}{2}} = 1.15$ apart from the original center $A_1^{(0)}$, $B_1^{(0)}$, $S_1^{(0)}$. The phaseshifts measured at the point u_1'', v_1'', w_1'' are indicated in Table V as solution III (non causal) ($\alpha_1 < 0$, $v_1'' < 0$) and solution IV (causal) ($\alpha_1 > 0$, $v_1'' > 0$). These solutions, which possess the two sets of P-phaseshifts coinciding, complete the analysis as far as the phaseshifts for $T = \frac{1}{2}$ state are concerned.

Using Eq. (7a) we have evaluated the new angular distribution coefficients for the scattering in the isotopic spin $\frac{3}{2}$. It is found $A_3' = 0.941$, $B_3' = 0.304$, $S_3' = 2.041$. From these new central values we have evaluated the new center of the ellipsoid in (u, v, w)-space, which is $u_3' = -0.081$, $v_3' = -1.232$, $w_3' = 0.118$. The following set of Fermi phaseshift for the $T = \frac{3}{2}$ state is thus found:

$$(9) \qquad \alpha_3' = -10.9^\circ \pm 4.7^\circ, \quad \alpha_{33}' = 97.5^\circ \pm 3.2^\circ, \quad \alpha_{31}' = -11.0^\circ \pm 5.2^\circ,$$

which is close to the set calculated by ANDERSON *et al.* From the four solutions previously determined we drop the two non causal solutions (I and III solutions) and the solution II, which give a very poor fit to the data. Therefore, the

TABLE V. – *Summary of the analysis of pion-nucleon scattering at* 189 MeV.

Solution	I (non causal)	II (causal)	III (non causal)	IV (causal)	ANDERSON *et al.*
α_1	— 13.6°	— 13.6°	— 17.5°	17.5°	— 2.8°
α_{13}	— 8.3°	8.3°	1.6°	— 1.6°	— 2.1°
α_{11}	— 8.3°	8.3°	1.6°	— 1.6°	— 2.6°
α_3	— 13.0°	— 13.0°	— 13.0°	— 13.0°	— 11.3°
α_{33}	97.5°	97.5°	97.5°	97.5°	98.8°
α_{31}	— 10.4°	— 10.4°	— 10.4°	— 10.4°	— 11.6°
A_3	0.955	0.955	0.955	0.955	0.917
B_3	0.394	0.394	0.394	0.394	0.345
C_3	3.281	3.281	3.281	3.281	3.345
A_-	0.110	0.154	0.176	0.120	0.107
B_-	0.008	0.013	0.027	0.055	0.067
C_-	0.587	0.420	0.353	0.387	0.409
A_0	0.245	0.201	0.202	0.258	0.204
B_0	— 0.002	0.007	— 0.051	0.044	0.056
C_0	0.632	0.799	0.745	0.712	0.712
M_+	0.07	0.07	0.07	0.07	0.27
M_-	11.44	8.03	16.53	1.17	1.29
M_0	0.32	1.11	0.63	1.27	2.10
$\mathcal{M}$	11.83	9.21	17.23	2.52	3.66
N_+	1.09	1.09	1.09	1.09	1.34
N_-	—	—	—	3.67	3.32
N_0	—	—	—	2.89	2.96
$\mathcal{N}$	—	—	—	7.65	7.62

selection of the best set of six phaseshifts must be performed between the following two

$$(10a)\qquad \begin{cases} \alpha_3 = -13.0^\circ, & \alpha_{33} = 97.5^\circ, & \alpha_{31} = -10.4^\circ, \\ \alpha_1 = 17.5^\circ, & \alpha_{13} = -1.6^\circ, & \alpha_{11} = -1.6^\circ, \end{cases}$$

$$(10b)\qquad \begin{cases} \alpha_3' = -10.9^\circ, & \alpha_{33}' = 97.5^\circ, & \alpha_{31}' = -11.0^\circ, \\ \alpha_1 = 17.5^\circ, & \alpha_{13} = -1.6^\circ, & \alpha_{11} = -1.6^\circ. \end{cases}$$

With the set (10*a*) it is found $\mathcal{T} = 7.65$ and $\mathcal{MT} = 2.52$, while the set (10*b*) gives $\mathcal{T} = 8.05$ and $\mathcal{MT} = 2.57$ ($M_+ = 0.46$, $M_- = 1.31$, $M_0 = 0.80$; $N_+ = 1.40$, $N_- = 4.46$, $N_0 = 2.19$). The criterion of minimum $\mathcal{T}$ suggests to choose the former set as the best one (see Table V).

It is interesting to investigate why the set (10*a*) and the ANDERSON set lead practically to the same value of $\mathcal{T}$. The distance of the point A_1'', B_1'', S_1'' from the original center $A_1^{(0)}$, $B_1^{(0)}$, $S_1^{(0)}$ is $M_1^{\frac{1}{2}} = 1.11$. The analyzer reproduces the Anderson set for $i = 1$ when its parameters are $u_1 = 3.984$, $v_1 = -0.332$, $w_1 = 1.995$, corresponding in (A, B, S)-space to the point $A_1 = 0.025$, $B_1 = 0.090$, $S_1 = 0.008$. This point lies a distance $M_1^{\frac{1}{2}} = 2.11$ from the experimental center $A_1^{(0)}$, $B_1^{(0)}$, $S_1^{(0)}$. The Anderson point is located on the opposite side of the point $A_1^{(0)}$, $B_1^{(0)}$, $S_1^{(0)}$ with respect to the origin of the (A, B, S)-space. In fact, the distance of the experimental center from the point $A_1 = B_1 = S_1 = 0$ (all phaseshifts zero for the $T = \frac{1}{2}$ state), is 1.51 only. Combining the set (10*a*) with all phaseshifts zero for $i = 1$ it is found $\mathcal{T} = 11.70$, whereas it is found $\mathcal{T} = 10.80$ using the Anderson set for the $T = \frac{3}{2}$ state together with all phaseshifts zero for the $T = \frac{1}{2}$ state. The fact that in the latter case the value of $\mathcal{T}$ is lower than in the former one means that the phaseshifts for the $T = \frac{3}{2}$ state have been sensibly distorted by the electronic computer in order to make them consistent with the small and negative values of the phaseshifts for $i = 1$. This can be easily seen by comparing the values of M_+ and N_-, calculated with the set (10*a*) and with the Anderson set (Table V). Furthermore, combining the Anderson set for the $T = \frac{3}{2}$ state with our set for the $T = \frac{1}{2}$ state it is found $\mathcal{T} = 8.14$, and this value confirms the previous argument. We believe that a slight improvement of the set (10*a*) will reduce the corresponding value of $\mathcal{T}$.

Appendix A

Phaseshift Variability within the Initial Part of the Compatibility Region.

The coordinates (u_i, v_i) of the terminal P_1 of the linkage system are very close to the point (4, 0) for low energy positive scattering (< 70 MeV) ($i = 3$) and for the scattering in the isotopic spin state $\frac{1}{2}$ ($i = 1$) up to about 200 MeV (Sect. 3). This important region of the (u, v)-plane (plane of the analyzer) has been systematically explored along some particular tracks. The corresponding phaseshifts are listed in Tables A, B, C, D, E. These values are of general validity in the sense that they are established only by the special kinematical properties of the mechanical analyzer, which are implied by S and P wave approximation and by the assumption of the charge independence of nuclear forces.

Table A. - *Values (in degrees) of the phaseshifts* α_i, $\alpha_{i3}^{(\pm)}$, $\alpha_{i4}^{(\mp)}$ *along the track* $3.8 \leqslant u_i \leqslant 4$, $v_i = 0$. The values of w_i, within the compatibility region, are chosen in the following way: 1) minimum value of w_i; the corresponding set of phaseshifts is indicated with (m). 2) maximum values of w_i (M); 3) value of w_i satisfying the condition $\alpha_i = 0$ (0); 4) value of w_i corresponding to the minimum value of α_i (a); 5) value of w_i corresponding to the maximum value of α_i (A); 6) value of w_i corresponding to $\operatorname{tg} \alpha_i = v_i/u_i$ (T). From the last column of the Table it is seen that the same set of phaseshifts is associated with different conditions imposed on w_i. The phaseshifts are measured to the nearest half degree and are listed according to decreasing values of α_i. Δu_i is the difference between the upper limit of u_i in the considered interval and its actual value (Track A, Fig. 1).

Δu_i	u_i	w_i	α_i	$\alpha_{i3}^{(-)}$	$\alpha_{i1}^{(-)}$	$\alpha_{i3}^{(+)}$	$\alpha_{i1}^{(+)}$	Conditions
0	4.000	2.000	0	0	0	0	0	m, M, O, a, A, T
0.020	3.980	1.985	4.9	— 1.5	— 1.5	— 1.5	— 1.5	M, A
		1.975	0.0	2.1	— 4.3	— 2.1	4.3	m, O, T
		1.985	— 4.9	1.5	1.5	1.5	1.5	M, a
0.050	3.950	1.962	7.9	— 2.5	— 2.5	— 2.5	— 2.5	M, A
		1.926	0.0	3.8	— 7.3	— 3.8	7.3	m, O, T
		1.962	— 7.9	2.5	2.6	2.5	2.5	M, a
0.100	3.900	1.925	11.2	— 3.6	— 3.6	— 3.6	— 3.6	M, A
		1.853	0.0	5.2	— 10.6	— 5.2	10.6	m, O, T
		1.925	— 11.2	3.6	3.6	3.7	3.6	M, a
0.200	3.800	1.848	16.0	— 5.0	— 5.0	— 5.0	— 5.0	M, A
		1.718	0.0	7.3	15.2	— 7.3	15.2	m, O, T
		1.848	— 16.0	5.0	5.0	5.0	5.0	M, a

TABLE B. – *Values (in degrees) of the phaseshifts* α_i, $\alpha_{i3}^{(\pm)}$, $\alpha_{i1}^{(\mp)}$ *along the track* $3.780 \leqslant u_i \leqslant 3.980$, $v_i = 0.4$ The values of w_i, within the compatibility region, are chosen according to the same conditions as in Table A (Track *B*, Fig. 1).

Δu_i	u_i	w_i	α_i	$\alpha_{i3}^{(-)}$	$\alpha_{i1}^{(-)}$	$\alpha_{i3}^{(+)}$	$\alpha_{i1}^{(+)}$	Conditions
0	3.980	1.995	2.8	2.8	2.8	2.8	2.8	*m, M, O, A, T*
0.020	3.960	1.964	7.7	1.2	1.2	1.2	1.2	*A*
		1.959	5.7	4.1	— 1.7	— 0.2	5.6	*m*
		1.969	2.8	5.0	— 1.4	0.6	7.1	*T*
		1.984	0.0	5.5	0.4	2.1	7.2	*O*
		1.998	— 2.0	4.6	4.6	4.6	4.6	*M, a*
0.050	3.930	1.928	10.9	0.2	0.2	0.2	0.2	*A*
		1.910	5.8	5.7	— 5.0	— 1.7	9.0	*m*
		1.923	2.9	6.5	— 4.4	— 0.7	10.2	*T*
		1.935	0.0	7.3	— 3.0	0.4	10.8	*O*
		1.985	— 5.0	5.5	5.5	5.5	5.5	*M, a*
0.100	3.880	1.880	14.1	— 0.7	— 0.7	— 0.7	— 0.7	*A*
		1.839	6.0	7.4	— 7.9	— 3.1	12.2	*m*
		1.845	2.9	8.3	— 7.4	— 2.4	13.3	*T*
		1.861	0.0	8.8	— 6.5	— 1.1	14.2	*O*
		1.957	— 8.4	6.3	6.3	6.3	6.3	*M, a*
0.200	3.780	1.786	19.1	— 2.0	— 2.0	— 2.0	— 2.0	*A*
		1.694	5.0	9.7	— 12.7	— 5.0	17.4	*m*
		1.702	3.0	10.6	— 11.9	— 4.4	18.1	*T*
		1.724	0.0	11.2	— 10.6	— 3.0	18.8	*O*
		1.898	— 13.0	8.3	8.3	8.3	8.3	*M, a*

TABLE C. – *Values (in degrees) of the phaseshifts* α_i, $\alpha_{i3}^{(\pm)}$, $\alpha_{i1}^{(\pm)}$ *along the track* $3.673 \leqslant u_i \leqslant 3.873$, $v_i = 1$. The values of w_i within the compatibility region are chosen according to the same conditions as in Table A (Track *C*, Fig. 1).

Δu_i	u_i	w_i	α_i	$\alpha_{i3}^{(-)}$	$\alpha_{i1}^{(-)}$	$\alpha_{i3}^{(+)}$	$\alpha_{i1}^{(+)}$	Conditions
0	3.873	1.968	7.2	7.2	7.2	7.2	7.2	*m, M, a, A, T*
0.020	3.853	1.906	12.7	5.6	5.6	5.6	5.6	*m, A*
		1.940	7.2	9.5	2.7	5.0	11.8	*T*
		1.997	2.0	8.9	8.9	8.9	8.9	*M, a*
0.050	3.823	1.862	15.2	4.7	4.7	4.7	4.7	*A*
		1.859	14.5	7.2	2.2	3.5	8.5	*m*
		1.894	7.3	10.9	— 0.1	3.7	14.7	*T*
		1.992	0.0	10.5	7.0	8.4	12.0	*O*
		2.000	— 0.5	9.8	9.8	9.8	9.8	*M, a*
0.100	3.773	1.800	18.4	3.8	3.8	3.8	3.8	*A*
		1.790	14.5	9.3	— 2.6	1.1	13.0	*m*
		1.823	7.4	12.6	— 2.9	2.4	17.9	*T*
		1.918	0.0	14.0	2.3	6.2	17.8	*O*
		1.992	— 3.6	10.9	10.9	10.9	10.9	*M, a*
0.200	3.673	1.682	23.5	2.7	2.7	2.7	2.7	*A*
		1.644	17.3	10.3	— 7.0	— 1.3	16.0	*m*
		1.680	7.6	15.0	— 7.1	0.5	22.6	*T*
		1.787	0.0	16.5	— 2.5	4.0	23.0	*O*
		1.958	— 8.3	12.5	12.5	12.5	12.5	*M, a*

TABLE D. – *Values (in degrees) of the phaseshifts* $\alpha_i = \alpha_{i3}^{(\pm)} = \alpha_{i1}^{(\pm)}$ *along the limiting track (L):* $u_i = [4^2 - v_1^2]^{\frac{1}{2}}$, $0 \leqslant v_i \leqslant 1.5$ *and related value of* w_i (Track *D*, Fig. 1).

u_i	v_i	w_i	$\alpha_i = \alpha_{i3}^{(\pm)} = \alpha_{i1}^{(\pm)}$	Conditions
4.000	0	2.000	0.0	*m, M, O, a, A, T*
4.000	0.050	2.000	0.3	*m, M, a, A, T*
3.999	0.100	2.000	0.7	*m, M, a, A, T*
3.997	0.150	1.999	1.1	*m, M, a, A, T*
3.995	0.200	1.999	1.4	*m, M, a, A, T*
3.989	0.300	1.997	2.1	*m, M, a, A, T*
3.980	0.400	1.995	2.8	*m, M, a, A, T*
3.955	0.600	1.989	4.3	*m, M, a, A, T*
3.873	1.000	1.968	7.2	*m, M, a, A, T*
3.708	1.500	1.927	11.0	*m, M, a, A, T*

TABLE E. – *Values (in degrees) of the phaseshifts* α_i, $\alpha_{i3}^{(\pm)}$, $\alpha_{i1}^{(\pm)}$ *along the track parallel to the limiting track* (L): $u_i = [4^2 - v_i^2]^{\frac{1}{2}} - 0.1$, $0 \leqslant u_i \leqslant 1.5$. The values of w_i, within the compatibility region, are chosen according to the same conditions as in Table A. (Track *E*, Fig. 1).

Δu_i	v_i	w_i	α_i	$\alpha_{i3}^{(-)}$	$\alpha_{i1}^{(-)}$	$\alpha_{i3}^{(+)}$	$\alpha_{i1}^{(+)}$	Conditions
3.900	0	1.925	11.2	— 3.6	— 3.6	— 3.6	— 3.6	*M*, *A*
		1.852	0.0	5.2	— 10.6	— 5.2	10.6	*m*, *O*, *T*
		1.925	— 11.2	3.6	3.6	3.6	3.6	*M*, *a*
3.900	0.050	1.918	11.6	— 3.2	— 3.2	— 3.2	— 3.2	*A*
		1.850	2.2	5.4	— 10.5	— 5.5	10.4	*m*
		1.851	0.3	5.6	— 10.2	— 5.0	10.8	*T*
		1.855	0.0	5.6	— 10.0	— 4.6	11.0	*O*
		1.927	— 11.0	4.0	4.0	4.0	4.0	*M*, *a*
3.899	0.100	1.913	12.0	— 3.0	— 3.0	— 3.0	— 3.0	*A*
		1.849	2.0	6.2	— 9.6	— 4.7	11.2	*m*
		1.851	0.7	6.2	— 9.6	— 4.5	11.3	*T*
		1.855	0.0	6.1	— 9.5	— 4.1	11.5	*O*
		1.933	— 10.5	4.1	4.1	4.1	4.1	*M*, *a*
3.895	0.200	1.903	12.7	— 2.2	— 2.2	— 2.2	— 2.2	*A*
		1.847	3.0	6.4	— 9.3	— 4.2	11.5	*m*
		1.851	1.4	6.9	— 8.9	— 3.9	11.9	*T*
		1.855	0.0	7.2	— 8.4	— 3.2	12.4	*O*
		1.941	— 9.8	5.0	5.0	5.0	5.0	*M*, *a*
3.889	0.300	1.889	13.6	— 1.4	— 1.4	— 1.4	— 1.4	*A*
		1.844	4.0	7.0	— 8.7	— 3.5	12.2	*m*
		1.849	2.2	8.3	— 7.4	— 2.4	13.3	*T*
		1.855	0.0	8.2	— 7.4	— 2.0	13.6	*O*
		1.951	— 9.0	12.0	12.0	12.0	12.0	*M*, *a*
3.880	0.400	1.880	14.1	— 0.7	— 0.7	— 0.7	— 0.7	*A*
		1.839	6.0	7.4	— 7.9	— 3.1	12.2	*m*
		1.854	2.9	8.3	— 7.4	— 2.4	13.3	*T*
		1.861	0.0	8.8	— 6.5	— 1.1	14.2	*O*
		1.957	— 8.4	6.3	6.3	6.3	6.3	*M*, *a*

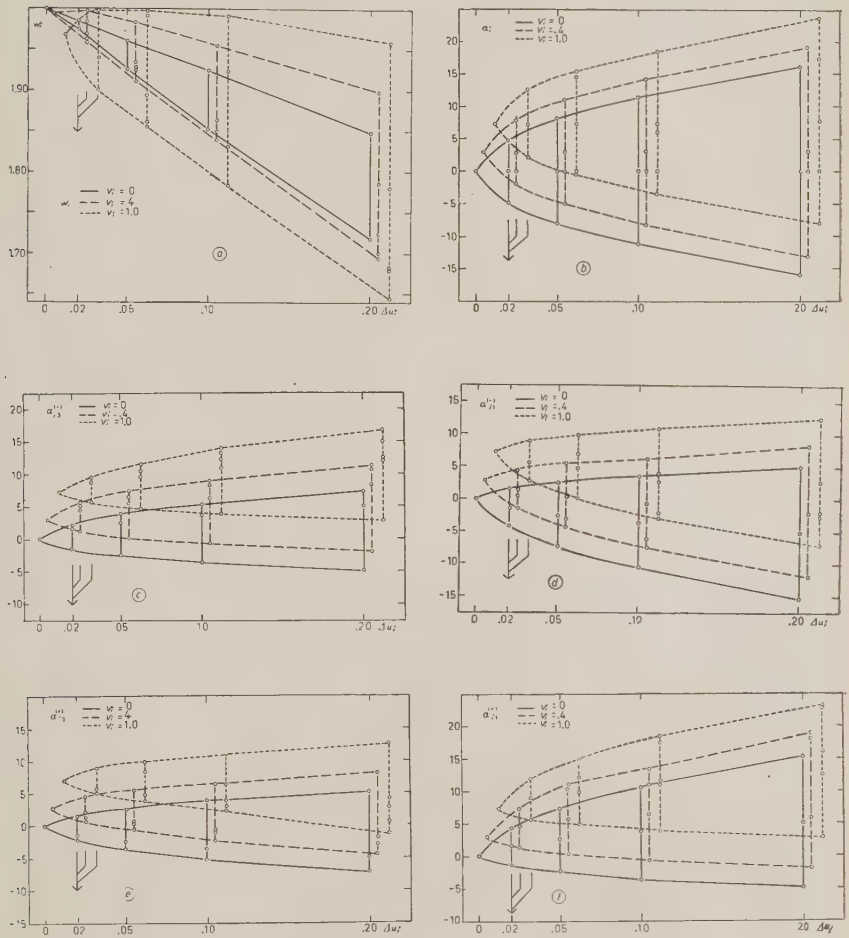

Fig. 6. – Limits of variability of *a*) w_i, *b*) α_i, *c*) $\alpha_{i3}^{(-)}$, *d*) $\alpha_{i1}^{(-)}$, *e*) $\alpha_{i3}^{(+)}$, *f* $\alpha_{i1}^{(+)}$, along the following tracks: I) $\mathbf{3.8} \leqslant \boldsymbol{u}_i \leqslant \mathbf{4}$, $\boldsymbol{v}_i - \mathbf{0}$ (full line, Table A); II) $\mathbf{3.780} \leqslant \boldsymbol{u}_i \leqslant \mathbf{3.980}$, $\boldsymbol{v}_i = \mathbf{0.4}$ (dotted line, Table B); III) $\mathbf{3.673} \leqslant \boldsymbol{u}_i \leqslant \mathbf{3.873}$, $\boldsymbol{v}_i = \mathbf{1}$ (dashed line, Table C). The abscissa is given as the difference Δu_i between the upper limit of u_i in the considered interval and its actual value. The marked values of w_i correspond to those considered in Tables A, B, C.

TABLE E (*continued*).

u_i	v_i	w_i	α_i	$\alpha_{i3}^{(-)}$	$\alpha_{i1}^{(-)}$	$\alpha_{i3}^{(+)}$	$\alpha_{i1}^{(+)}$	Conditions
3.855	0.600	1.853	15.7	0.7	0.7	0.7	0.7	*A*
		1.820	8.7	8.1	— 6.9	— 2.0	13.0	*m*
		1.843	4.3	9.7	— 6.0	— 0.9	14.8	*T*
		1.873	0.0	10.8	— 3.8	1.1	15.7	*O*
		1.971	— 6.9	8.0	8.0	8.0	8.0	*A*, *m*
3.773	1.000	1.800	18.4	3.8	3.8	3.8	3.8	*A*
		1.790	14.5	9.3	— 2.6	1.1	13.0	*m*
		1.823	7.4	12.6	— 2.9	2.4	17.9	*T*
		1.918	0.0	14.0	2.3	6.1	17.8	*O*
		1.992	— 3.6	10.9	10.9	10.9	10.9	*M*, *a*
3.608	1.500	1.718	22.0	7.5	7.5	7.5	7 .5	*m*, *A*
		1.788	11.2	16.2	1.1	6.0	21.2	*T*
		2.000	0.5	14.7	14.7	14.7	14.7	*M*, *a*

The upper and lower limits of w_i and of the associated phaseshifts α_i, $\alpha_{i3}^{(\pm)}$, $\alpha_{i1}^{(\pm)}$ along the tracks considered in Tables A, B, C are represented versus Δu_i in Fig. 6. The limiting values of w_i and those of the associated phaseshifts α, $\alpha_{i3}^{(\mp)}$, $\alpha_{i1}^{(\pm)}$ along the tracks considered in Table D and E are plotted versus r_i in Fig. 7 respectively 8.

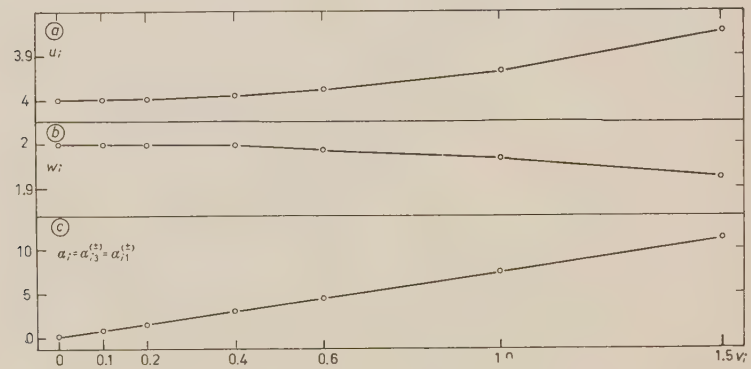

Fig. 7. – *a*) values of u_i, *b*) values of w_i and *c*) values (in degrees) of the phaseshifts $\alpha_i = \alpha_{i3}^{(\pm)} = \alpha_{i1}^{(\pm)}$ versus v_i along the track $\boldsymbol{u}_i = [\mathbf{4}^2 - \boldsymbol{v}_i^2]^{\frac{1}{2}}$ (Table D).

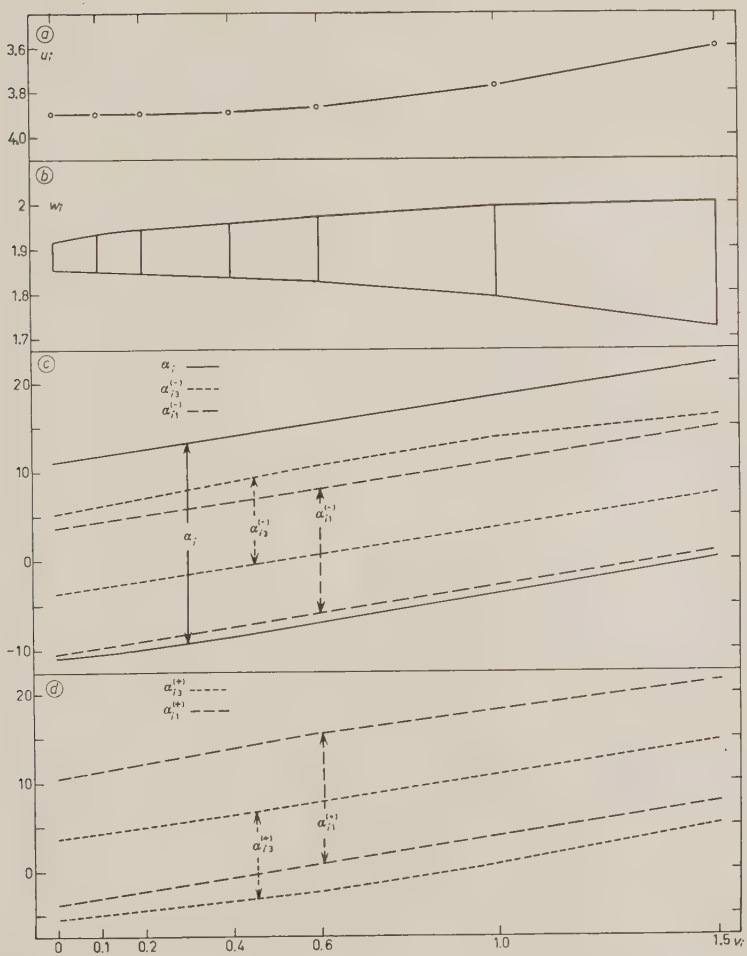

Fig. 8. – *a*) values of u_i, *b*) limits of variability of w_i, limits of variability of the phaseshifts, *c*) α_i, $\alpha_{i3}^{(-)}$, $\alpha_{i1}^{(-)}$ and *d*) $\alpha_{i3}^{(+)}$, $\alpha_{i1}^{(+)}$ along the track $\boldsymbol{u}_i=[\mathbf{4}^2-\boldsymbol{v}_i^2)^{\frac{1}{2}}-\mathbf{0.1}$ versus v_i (Table E).

Appendix B

Input Parameters of the Analyzer and Angular Distribution Coefficients along some Tracks on the Boundary of the Compatibility Region.

On the basis of the preceeding Tables and diagrams it is easy to obtain, by interpolation and without using the analyzer, several information on the possible sets of phaseshifts and to check the internal consistency of any solution, corresponding to a scattering situation which implies the analyzer input parameters u_i, v_i, w_i to be close to the limiting point (4, 0, 2). The correlation between u_i, v_i and the limits of w_i, given in Tables A, B, C, D, E, is nothing but the numerical solution, obtained mechanically, of the equation $w_i = F(u_i, v_i)$, discussed in Sect. **1**. A panoramic view of the compatibility region, described by this equation, together with the tracks explored by the linkage system, is given in Fig. 1.

To make as simple as possible the search for the point A_1', B_1', S_1' on the boundary surface of the compatibility region in the (A, B, S)-space, we have transformed the numerical solutions of the equation $w_i = F(u_i, v_i)$ into the numerical solution of the equation $A_i = \Phi(B_i, S_i)$ (Sect. **1**). The coordinates of some points of the explored tracks in (u, v, w)-space and in (A, B, S)-space are listed in the following Tables. Except for the limiting track (L), all other Tables have been derived from the Tables given in Appendix A, by associating to the points of the considered tracks in the plane of the analyzer, the minimum (m) and the maximum (M) value of w_i. The diagrams of the projections of the compatibility region on the co-ordinate planes can easily be drawn using these Tables.

Table A(m). – *Input parameters u_i, v_i, w_i and angular distribution coefficients A_i, B_i, C_i and S_i along the track* $\mathbf{3.7 \leqslant u_i \leqslant 4}$, $\mathbf{v_i = 0}$, $\mathbf{w_i =}$ **minimum**.
(Track A(m), Fig. 1).

u_i	v_i	w_i	A_i	B_i	C_i	S_i
4.000	0	2.000	0	0	0	0
3.980	0	1.975	0.013	— 0.005	— 0.009	0.010
3.950	0	1.926	0.037	0.000	— 0.036	0.025
3.900	0	1.852	0.074	0.001	— 0.072	0.050
3.800	0	1.718	0.140	— 0.008	— 0.120	0.100
3.700	0	1.571	0.214	0.001	— 0.192	0.150

TABLE A(M). - *Input parameters* u_i, v_i, w_i *and angular distribution coefficients* A_i, B_i, C_i *and* S_i *along the track* $3.7 \leqslant u_i \leqslant 4$, $v_i = 0$, $w_i =$ **maximum**. (Track $A(M)$, Fig. 1).

u_i	v_i	w_i	A_i	B_i	C_i	S_i
4.000	0	2.000	0.	0	0	0
3.980	0	1.985	0.007	— 0.015	0.009	0.010
3.950	0	1.962	0.019	— 0.036	0.018	0.025
3.900	0	1.925	0.077	— 0.072	0.039	0.050
3.800	0	1.848	0.076	— 0.138	0.072	0.100
3.700	0	1.777	0.111	— 0.205	0.117	0.150

TABLE B(m). - *Input parameters* u_i, v_i, w_i *and angular distribution coefficients* A_i, B_i, C_i *and* S_i *along the track* $3.780 \leqslant u_i \leqslant 3.980$, $v_i = 0.4$, $w_i =$ **minimum**. (Track $B(m)$, Fig. 1).

u_i	v_i	w_i	A_i	B_i	C_i	S_i
3.980	0.400	1.995	0.002	0.015	0.024	0.010
3.960	0.400	1.959	0.020	0.021	0.000	0.020
3.930	0.400	1.910	0.045	0.026	— 0.030	0.035
3.880	0.400	1.839	0.090	0.026	— 0.063	0.060
3.780	0.400	1.694	0.153	0.028	— 0.129	0.110

TABLE B(M). - *Input parameters* u_i, v_i, w_i *and angular distribution coefficients* A_i, B_i, C_i *and* S_i *along the track* $3.780 \leqslant u_i \leqslant 3.980$, $v_i = 0.4$, $w_i =$ **maximum**. (Track $B(M)$, Fig. 1).

u_i	v_i	w_i	A_i	B_i	C_i	S_i
3.980	0.400	1.995	0.002	0.015	0.024	0.010
3.960	0.400	1.998	0.001	0.014	0.057	0.020
3.930	0.400	1.985	0.008	— 0.048	0.081	0.035
3.880	0.400	1.958	0.021	— 0.094	0.117	0.060
3.780	0.400	1.898	0.051	— 0.176	0.177	0.110

TABLE C(m). - *Input parameters* u_i, v_i, w_i *and angular distribution coefficients* A_i, B_i, C_i *and* S_i *along the track* $3.673 \leqslant u_i \leqslant 3.783$, $v_i = 1$, $w_i =$ **minimum**. (Track $C(m)$, Fig. 1).

u_i	v_i	w_i	A_i	B_i	C_i	S_i
3.783	1.000	1.968	0.016	0.094	0.142	0.063
3.853	1.000	1.906	0.047	0.129	0.078	0.073
3.823	1.000	1.859	0.070	0.133	0.054	0.088
3.773	1.000	1.790	0.105	0.133	0.024	0.113
3.673	1.000	1.718	0.178	0.142	— 0.045	0.163

TABLE C(M). – *Input parameters* u_i, v_i, w_i *and angular distribution coefficients* A_i, B_i, C_i *and* S_i *along the track* $3.673 \leqslant u_i \leqslant 3.873$, $v_i = 1$, $w_i =$ **maximum**. (Track $C(M)$, Fig. 1),

u_i	v_i	w_i	A_i	B_i	C_i	S_i
3.783	1.000	1.968	0.016	0.095	0.141	0.063
3.853	1.000	1.997	0.001	0.038	0.216	0.073
3.823	1.000	2.000	0.000	— 0.023	0.264	0.088
3.773	1.000	1.992	0.004	— 0.069	0.327	0.113
3.673	1.000	1.958	0.021	— 0.172	0.426	0.163

TABLE D(L). – *Input parameters* u_i, v_i, w_i *and angular distribution coefficients* A_i, B_i C_i *and* S_i *along the* limiting track $u_i^2 + v_i^2 = 4$. (Track $D(L)$, Fig. 1).

u_i	v_i	w_i	A_i	B_i	C_i	S_i
4.000	0	2.000	0	0	0	0
3.999	0.100	2.000	0	0.001	0.003	0.001
3.995	0.200	1.999	0.001	0.004	0.003	0.002
3.989	0.300	1.997	0.001	0.009	0.012	0.005
3.980	0.400	1.995	0.002	0.015	0.024	0.010
3.955	0.600	1.989	0.005	0.034	0.051	0.022
3.873	1.000	1.968	0.016	0.095	0.141	0.063
3.708	1.500	1.927	0.036	0.218	0.348	0.146

TABLE E(m). *Input parameters* u_i, v_i, w_i *and angular distribution coefficients* A_i, B_i, C_i *and* S_i *along the track* $u_i = [4^2 - v_i^2]^{\frac{1}{2}} - 0.1$, $0 \leqslant v_i \leqslant 1.5$, $w_i =$ **minimum**. (Track $E(m)$, Fig. 1).

u_i	v_i	w_i	A_i	B_i	C_i	S_i
3.900	0	1.852	0.074	0.001	— 0.072	0.050
3.899	0.100	1.849	0.075	0.004	— 0.072	0.051
3.895	0.200	1.847	0.076	0.008	— 0.072	0.052
3.889	0.300	1.844	0.078	0.015	— 0.069	0.055
3.880	0.400	1.839	0.081	0.025	— 0.063	0.060
3.855	0.600	1.820	0.090	0.057	— 0.051	0.073
3.773	1.000	1.790	0.105	0.133	0.024	0.113
3.608	1.500	1.718	0.141	0.295	0.165	0.196

TABLE E(m). – *Input parameters* u_i, v_i, w_i *and angular distribution coefficients* A_i, B_i, C_i *and* S_i *along the track* $\boldsymbol{u_i=[4^2-v_i^2]^{\frac{1}{2}}-0.1,\ 0\leqslant v_i\leqslant 1.5,\ w_i=\text{maximun}}$.
(Track $E(M)$, Fig. 1).

u_i	v_i	w_i	A_i	B_i	C_i	S_i
3.900	0	1.925	0.038	— 0.072	0.036	0.050
3.899	0.100	1.933	0.033	— 0.077	0.054	0.051
3.895	0.200	1.941	0.029	— 0.086	0.069	0.052
3.889	0.300	1.951	0.024	— 0.095	0.093	0.055
3.880	0.400	1.957	0.021	— 0.094	0.114	0.060
3.855	0.600	1.971	0.015	— 0.092	0.171	0.072
3.773	1.000	1.992	0.004	— 0.069	0.327	0.113
3.608	1.500	2.000	0.000	0.013	0.588	0.196

SUPPLEMENTO AL VOLUME III, SERIE X
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Relativistic Quantum Mechanics and Mass-Quantization.

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CONTENTS. — Introduction. – **1.** Center of mass in space-time. – **2.** Wave equations in relativistic quantum mechanics. – **3.** Derivation of the self-potentials for pseudoscalar mesons. – **4.** Evaluation of mass eigenvalues in terms of power series expansions. – **5.** Variational method. – **6.** Concluding remarks.

Introduction.

It is well known that, as a result of the independent investigations of TOMONAGA, SCHWINGER and FEYNMAN, the treatment of the difficulties inherent in the quantum theory of fields has made notable advances. Using the method of renormalization, they arrived at finite results which are free from the divergence difficulties. In their formulations the troubles which one had from the very beginning of the quantum field theory have been at least partially overcome. Nevertheless, it is not at all clear that the divergences can be made finite by some physical and mathematical manipulations.

In the last few years, various mathematical formalisms for the treatment of such a problem in connection with the quantization of mass, have been proposed by many authors [1]. In their papers so far published, emphasis has been placed on the prediction of a mass spectrum of elementary particles under

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[1] H. YUKAWA: *Phys. Rev.*, **77** 219 (1950); **80**, 1047 (1950); **91**, 415, 416 (1953); A. PAIS: *Progr. Theor. Phys.*, **10**, 457 (1953); *Proc. Nat. Acad. Sci.*, **40**, 484, 835 (1954); W. HEISENBERG: *Nachr. Göttinger Akad. Wiss.*, 111 (1953); *Zeits. f. Naturfors.*, **9***a*, 292 (1954) **10***a*, 425 (1955); O. HARA, T. MARUMORI, Y. OHNUKI and H. SHIMODAIRA: *Progr. Theor. Phys.*, **12**, 177 (1954); J. RAYSKI: *Nuovo Cimento*, **10**, 1729 (1953); **2**, 255 (1955).

some assumptions about the interaction and internal structure of the particles. There are, however, some points that may be worthwhile to discuss. The theory of mass spectrum should be formulated to remove the divergences associated with mass corrections, due to the self-interaction of particles, and at the same time to give observed mass values for the particles. Ordinarily, these two features are considered separately. Most theories so far proposed have not related in a comprehensive and intelligible manner the so-called divergent mass corrections to the experimental mass values of the particles. Furthermore, since the models and interactions adopted there are not so familiar, it is not easy to see whether the theories give us mass spectra which can be compatible with experimental results.

In two previous papers [2] we have so far developed a theory of mass-quantization for the case of a Dirac particle interacting with a scalar photon. It was an especially simple example to deal with. No application to the case of an actual particle was given there, because our discussion of the mass problem was intentionally only a sketch. The most important idea mentioned in our previous papers I and II is that the divergent self-energy of the particle is not a serious difficulty of the quantum field theory but it may be considered to be a self-potential or self-action at a distance which is related closely to the quantization of the mass of particles. The previous treatment is, however, not completely satisfactory in some aspects. First, the relation between our theory and the usual theory of renormalization has not been analyzed. Secondly, there is an ambiguity in introducing the so-called internal co-ordinates into the theory as well as the fact that the center of mass can not be defined uniquely in the theory of relativity. Thirdly, it is not obvious that, when our method is applied to actual particles, it gives us a reasonable result concerning the mass spectrum of the particles.

The aim of this paper is to carry out in detail the program of mass-quantization for the case of pseudoscalar mesons interacting with nucleons through the usual couplings. In this paper the ambiguous points mentioned above will be considered more fully. Although our basic idea is not changed, some aspects of our previous discussion will be modified in a manner which is much more familiar.

We first discuss the definition of the center of mass of a system in relativistic particle dynamics. Under some conditions, we show how the concept of center of mass in the three dimensional case can be directly extended to a system in space-time. We then introduce the concept of external co-ordinates and internal co-ordinates for the one-body problem. We define a proper time parameter for each particle. Using these parameters, the wave equations for

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[2] H. ENATSU: *Progr. Theor. Phys.*, **11**, 125 (1954); **12**, 363 (1954). these papers will be referred to as I and II.

the particles and the system are set up in a way which is similar to that of Dirac, Fock and Podolsky's many-time theory. As an illustration, we consider the one-body problem of a pseudoscalar meson. We derive the self-potential of the meson which is expressed in terms of the space-like distance in the internal co-ordinates space. The self-potential is put into the wave equation which describes the internal motion of the meson. The wave equation is solved under a boundary condition in the usual way. The eigenvalues of mass obtained are compared with those of π-mesons and heavy mesons.

It is the purpose of this paper to emphasize how the Hamiltonian formalism which is a powerful method in the elementary quantum mechanics, can be applied to include self-interactions which are closely related to the quantized masses of elementary particles.

1. – Center of Mass in Space-Time.

The starting point of our analysis is a definition of the center of mass in space-time. The conventional treatment of the two-body problem in field theory involves the assumption that the co-ordinates of the center of mass go over into the normal definition of it in the non-relativistic limit. As is well known, there is an arbitrariness in the definition of the center of mass in the theory of relativity ([3]). It is felt that it might be worthwhile to consider this problem from the view-point of relativistic particle dynamics. For simplicity, let us suppose that there are two free particles in space-time, and their co-ordinates, momenta and masses are x'_μ, x''_μ, p'_μ, p''_μ, m_1 and m_2 ($\mu = 1, 2, 3, 4$) (*), respectively. Of course, the following discussion can be extended to more general cases. We transform x'_μ and x''_μ into new co-ordinates by a canonical transformation. The new co-ordinates vectors denote the position of the center of mass of the system and the relative distance between two particles. It should then, in principle, be possible to define a Hamiltonian formalism for the system. We assume that we can consider a *generalized Hamiltonian* for the system, which is shown later on, and we choose a common time τ for the two particles. The common time may be a proper time in the sense of relativity, or may be an absolute time somewhat analogous to that of non-relativistic mechanics. In the following discussion we assume implicitly the existence of such a time (+).

([3]) See attempts to define the center of mass in space-time: M. Born and K. Fuchs: *Nature*, **145**, 587 (1940); E. E. Salpeter and H. A. Bethe: *Phys. Rev.*, **84**, 1232 (1951); C. Møller: *Communications of the Dublin Institute for Advanced Studies*, ser. A, No. 5 (1949).

(*) $x'_4 = ix'_0$, $p'_4 = ip'_0$, etc.

(+) If $(x'_\mu - x''_\mu)^2 > 0$, we can easily define the common time for the particles.

Now we consider the transformations

$$\text{(1.1)} \qquad \begin{cases} x'_\mu\,, & x''_\mu \to x_\mu\,, & y_\mu \\ p'_\mu\,, & p''_\mu \to P_\mu\,, & p_\mu \end{cases}$$

where P_μ and p_μ are total and relative momenta, and x_μ and y_μ are the co-ordinates of the center of mass and the relative co-ordinates respectively. According to YUKAWA (4), it is convenient to imagine that for the one-body problem x_μ indicate external co-ordinates and y_μ stand for the internal co-ordinates of the particle considered. We define the relative co-ordinates and total momentum as follows:

$$\text{(1.2)} \qquad y_\mu = x'_\mu - x''_\mu\,,$$

and

$$\text{(1.3)} \qquad P_\mu = p'_\mu + p''_\mu\,,$$

where p'_μ and p''_μ satisfy the relations

$$\text{(1.4)} \qquad p'^2_\mu = -m_1^2\,, \qquad p''^2_\mu = -m_2^2\,.$$

The canonical conjugate variables to y_μ and P_μ are found to be

$$\text{(1.5)} \qquad p_\mu = (1-\alpha)p'_\mu - \alpha p''_\mu\,,$$

and

$$\text{(1.6)} \qquad x_\mu = \alpha x'_\mu + (1-\alpha)x''_\mu\,,$$

where α is an arbitrary parameter. In order to fix the parameter α, we define the *generalized Hamiltonian* of the system as (5)

$$\text{(1.7)} \qquad \mathfrak{M}_1 = \frac{p'^2_\mu}{m_1} + \frac{p''^2_\mu}{m_2} = -(m_1 + m_2)\,.$$

The Hamiltonian $\mathfrak{M}_1$ is not the fourth component of an energy-momentum vector, but a world scalar or the mass itself. We assume that after the transformation

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(4) H. YUKAWA: reference (1). The necessity of the introduction of external co-ordinates (external particle) and internal coordinates (internal particle) is discussed in detail by S. A. EDDINGTON: *Fundamental Theory* (Cambridge, 1946). This was kindly pointed out to the author by Dr. DESER.

(5) G. FALK: *Zeits. f. Phys.*, **132**, 44 (1952).

the Hamiltonian has the form

$$\mathfrak{M}_1 = \frac{P_\mu^2}{m} + \frac{p_\mu^2}{n}, \tag{1.8}$$

where m and n are constants which are determined uniquely by comparing the coefficient of $p_\mu'^2$ and $p_\mu''^2$ of (1.7) with those of (1.8), in which a replacement $P_\mu,\ p_\mu \to p_\mu',\ p_\mu''$ is made according to (1.3) and (1.5). We find, as a result of this comparison,

$$\frac{1}{m} + \frac{(1-\alpha)^2}{n} = \frac{1}{m_1}, \tag{1.9}$$

$$\frac{1}{m} + \frac{\alpha^2}{n} = \frac{1}{m_2}, \tag{1.10}$$

and

$$\frac{1}{m} = \frac{\alpha(1-\alpha)}{n}. \tag{1.11}$$

The solutions of these equations are

$$m = m_1 + m_2, \tag{1.12}$$

$$\alpha = \frac{m_1}{m_1 + m_2} = \frac{m_1}{m}, \tag{1.13}$$

and

$$n = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{m}. \tag{1.14}$$

Accordingly, it is natural to define the center of mass as in the case of non-relativistic mechanics. It should be noted that the energy-momentum vectors P_μ and p_μ obey the law of relativity, so that P_μ^2 and p_μ^2 must be constants. In order to define the center of mass in such a way that it has actually the total mass of the system, it is necessary to add to the relations mentioned above a supplementary condition which restricts the admissible freedoms of the vectors p_μ' and p_μ''; we have the relation

$$P_\mu^2 = -(m_1 + m_2)^2 + 2(m_1 m_2 + p_\mu' p_\mu''), \tag{1.15}$$

which is an immediate consequence of the relations (1.3) and (1.4). Then the supplementary condition is

$$(p'p'') = -m_1 m_2, \tag{1.16}$$

where $(p'p'')$ means $p_\mu' p_\mu''$.

The condition (1.16) now assures us that

$$p_\mu^2 = [(1-\alpha)p_\mu' - \alpha p_\mu'']^2 \tag{1.17}$$

$$= -2\left(\frac{m_1 m_2}{m^2}\right)(m_1 m_2 + p_\mu' p_\mu'') = 0\,,$$

in virtue of (1.4), (1.5) and (1.13). Thus we can get the relations

$$P_\mu^2 = -(m_1+m_2)^2 = -m^2\,, \qquad p_\mu^2 = 0. \tag{1.18}$$

As a remark, we observe that the condition (1.16) is not satisfied in the case when either of the masses m_1 or m_2 is zero. In this case, (1.16) implies that the scalar product of p_μ' and p_μ'' is zero. One of the energy-momentum vectors p_μ' or p_μ'' is a null vector, and the other is a time-like vector. According to the theory of relativity, we can prove that a null vector can not be orthogonal to a time-like vector. Therefore we exclude such a case.

The Hamiltonian $\mathfrak{M}_1$ given by (1.7) is not the only one compatible with the requirement that the center of mass of the system is defined in a relativistic invariant way. In order to consider the problem of the self-energy of the particles, it is convenient to introduce another form of Hamiltonian,

$$\mathfrak{M}_2 = 2(p_\mu'^2 + p_\mu''^2) = 2\left(\frac{P_\mu^2}{\beta} + \frac{p_\mu^2}{\gamma}\right), \tag{1.19}$$

where the factor 2 in front of the brackets is simply to make the definition of $\mathfrak{M}_2$ agree with that of an actual particle, and β and γ are parameters. By making use of the relations (1.3) and (1.5), we can determine the parameters α, β and γ as in the case of $\mathfrak{M}_1$. They are found to be

$$\alpha = \tfrac{1}{2}\,, \qquad \beta = 2\,, \qquad \gamma = \tfrac{1}{2}\,. \tag{1.20}$$

We see that these parameters are independent of m_1 and m_2. This is not the case for $\mathfrak{M}_1$.

The magnitudes of P_μ and p_μ are

$$P_\mu^2 = -(m_1+m_2)^2 + 2(m_1 m_2 + p_\mu' p_\mu'')\,, \tag{1.21}$$

and

$$p_\mu^2 = -\tfrac{1}{4}(m_1^2 + m_2^2 + 2p_\mu' p_\mu'')\,. \tag{1.22}$$

If we assume the same supplementary condition as that of (1.16), we find

$$P_\mu^2 = -(m_1 + m_2)^2 , \tag{1.23}$$

and

$$p_\mu^2 = -\tfrac{1}{4}(m_1 - m_2)^2 . \tag{1.24}$$

Then the Hamiltonian $\mathfrak{M}_2$ is written as

$$\mathfrak{M}_2 = 2(p_\mu'^2 + p_\mu''^2) = P_\mu^2 + 4p_\mu^2 = -2(m_1^2 + m_2^2) . \tag{1.25}$$

If one of the masses is zero, we can not apply these formulas to the system.

In this way, a classical system is described equally well by specifying the vectors (x_μ', p_μ') and (x_μ'', p_μ''), or by specifying the vectors (x_μ, P_μ) and (y_μ, p_μ) under the condition (1.16). There may be other possibilities of describing the system by considering more complicated Hamiltonians. The Hamiltonian $\mathfrak{M}_2$ is, however, most convenient for the problem which we are going to deal with. We desire eventually to discuss the problem of mass-quantization by using the $\mathfrak{M}_2$.

We wish next to consider the case of a quantum-mechanical system. We assume that the formulas mentioned above hold *symbolically* in relativistic quantum mechanics. This means that in a representation in which co-ordinates are diagonal, momenta are differential operators, and for a stationary state the masses of particles are numerical values.

We now introduce our first postulate concerning the center of mass both for a system and for an *elementary particle*. The center of mass has the value of the actual mass which is observed. For the system of particles we could consider this postulate as simply implying that the observed mass of a system is the total sum of the masses of all particles involved and the change of the mass arising from mutual interactions. To interpret the postulate in the case of an *elementary particle*, let us imagine that there is a particle having an observed mass m in space-time. Then the precise meaning of the postulate is that the variables denoting the position of the particle are x_μ which are defined by (1.6), and the energy-momentum vectors satisfy the relation (1.18) (or (1.23)). The observed mass m consists of two parts m_1 and m_2, namely $m = m_1 + m_2$, where m_1 and m_2 are the masses of points x_μ' and x_μ'' respectively. We use the word « points » instead of « particles », because we suppose that the mass points m_1 and m_2 are not actual particles, but somehow mathematical points which are necessary to describe the internal structure of the particle and are required for the calculation of the mass spectrum.

We shall illustrate the postulate by applying it to the case of two particles in a mutual potential. We represent the masses of free particles by m_1 and m_2.

Let us suppose that the potential is a function only of y_μ and the sign of it is negative, that is, $-V(y)$, $(V(y) > 0)$. Since there is an interaction, the masses of the particles are considered to become $m_1 - \delta m_1$ and $m_2 - \delta m_2$ respectively. If we limit our discussion to the stationary state of mass, we get the quantum-mechanical Hamiltonian $\mathfrak{M}_1$ as follows:

$$\mathfrak{M}_1 = \frac{p_\mu'^2}{(m_1 - \delta m_1)} + \frac{p_\mu''^2}{(m_2 - \delta m_2)} - V(y)\,, \tag{1.26}$$

$$= \frac{P_\mu^2}{(m_0 - \delta m)} + \frac{p_\mu^2}{n'} - V(y)\,, \tag{1.27}$$

where

$$\delta m = \delta m_1 + \delta m_2\,, \qquad m_0 = m_1 + m_2\,, \tag{1.28}$$

$$n' = \frac{(m_1 - \delta m_1)(m_2 - \delta m_2)}{m_0 - \delta m}\,, \tag{1.29}$$

$$p_\mu'^2 = -(m_1 - \delta m_1)^2\,, \qquad p_\mu''^2 = -(m_2 - \delta m_2)^2\,, \tag{1.30}$$

$$P_\mu^2 = -(m_0 - \delta m)^2\,, \qquad p_\mu^2 = 0\,. \tag{1.31}$$

The sum δm appears to be a total mass defect of the system as a result of the interaction. The division of δm into two parts δm_1 and δm_2 is not uniquely defined. However, in a practical problem it is not necessary to know δm_1 and δm_2 separately. The ultimate aim is to obtain δm as a function of coupling constants and other parameters. For that purpose we need only to assume that the eigenvalue of the Hamiltonian $\mathfrak{M}_1$ is of the form

$$-(m_0 - \delta m) - \delta m\,, \qquad (m_0 \geqslant 0,\ \delta m \geqslant 0), \tag{1.32}$$

and the actual mass m is the sum of a bare mass m_0 and a mass correction $-\delta m$. The first term of (1.32) shows the observed mass of the system. The last term is supposed to be the mass correction arising from the potential in the relative coordinate space. It is apparent that the mass correction is $-\delta m$ since the kinetic part of the Hamiltonian of the y_μ-space is always zero in virtue of the last equation of (1.31).

For the one-body problem, it is convenient to use the Hamiltonian $\mathfrak{M}_2$ given by (1.19). As was mentioned before, an *elementary particle* having a physical mass m may be supposed to consist of two mass parts m_1 and m_2, $(m = m_1 + m_2)$. There is an arbitrariness in the choice of the masses m_1 and m_2. The first and simplest method is to consider that the m can be divided into two equal parts. A second method is to imagine that one of the parts has a

bare mass m_0 and the other possesses a mass $-\delta m$. The first method to which we turn our attention gives rise to the following symbolic expressions of the Hamiltonian,

$$\mathfrak{M}_2 = 2(p_\mu'^2 + p_\mu''^2) - V(y)\,, \tag{1.33}$$

$$= P_\mu^2 + 4p_\mu^2 - V(y)\,, \tag{1.34}$$

$$= -m^2 - \delta m^2\,, \tag{1.35}$$

where

$$p_\mu'^2 = -\frac{m^2}{4}\,, \qquad p_\mu''^2 = -\frac{m^2}{4}\,, \tag{1.36}$$

$$P_\mu^2 = -m^2\,, \qquad p_\mu^2 = 0\,, \tag{1.37}$$

$$m = m_1 + m_2 = \frac{m}{2} + \frac{m}{2} (= m_0 - \delta m)\,. \tag{1.38}$$

The $-\delta m^2$ in (1.35) denotes the change of the mass due to the *self-potential* $-V(y)$ which is usually called the *self-energy of the particle*. We will discuss this in connection with the usual procedure of renormalization in the succeeding section. In what follows we shall consider only the eigenvalue problem of mass. This means that we assume that the self-potential has to be always *negative* or the self-energy of the particle has *negative* divergences.

When one employs the second method of division, one must examine more carefully the supplementary condition (1.16). The situation in the present case differs from that in other cases by the fact that one of the mass points has a negative mass $-\delta m$. Then it follows from (1.16) that

$$(p'p'') = m_0 \delta m\,. \tag{1.39}$$

We assume that the bare mass m_0 of the particle is always positive. Consequently, the relation (1.39) implies that the scalar product of two time-like vectors p_μ' and p_μ'' is positive. This is not the case for arbitrary time-like vectors. In order to satisfy this condition, we assume that the components of the vector p_μ'' are practically limited to negative values, that is, p_{μ_i}'' is a negative energy momentum vector, while all components of the vector p_μ' have positive values. Then we can obtain the symbolic expressions for the Hamiltonian,

$$\mathfrak{M}_2' = 2(p_\mu'^2 + p_\mu''^2) - V(y)\,, \tag{1.40}$$

$$= P_\mu^2 + 4p_\mu^2 - V(y)\,, \tag{1.41}$$

$$= -2(m_0^2 + \delta m^2) - \delta m^2\,, \tag{1.42}$$

where

$$p'^2_\mu = -m_0^2\,, \qquad p''^2_\mu = -\delta m^2\,, \tag{1.43}$$

$$P^2_\mu = -m^2\,, \qquad p^2_\mu = -(m_0+\delta m)^2\,, \tag{1.44}$$

$$m = m_0 - \delta m\,. \tag{1.45}$$

It should be noted that in this case the magnitude of the internal energy-momentum vector is different from zero.

So far we have investigated the possibility of defining the center of mass under some conditions both for the two-particle problem and for the one-body problem, and at the same time of getting the symbolic representations of the generalized quantum mechanical Hamiltonians. They will serve as prototypes enabling us to write down the corresponding Hamiltonians for the systems involving many particles and potentials. The discussion we have made in this section is of quite general validity.

2. – Wave Equations in Relativistic Quantum Mechanics.

We wish next to set up wave equations in space-time on the basis of the results obtained in the previous section.

Suppose that there are n free particles in space-time. First of all, we consider the proper times of the particles. Let us call the proper time in the theory of relativity the *classical proper time*. The proper time which we are going to introduce into relativistic quantum mechanics will be called simply the *proper time*. We assume that all particles have their proper times τ_i $(i = 1, 2, \ldots, n)$. When we consider a system of particles, we also assign a proper time τ to the center of mass of the system. Furthermore, we assume that the proper times τ_i and τ depend on the masses of the particles, and on the form of the Hamiltonian. However, it may happen that the proper times are independent of the masses as in the case of the classical proper time. In order to compare the proper times with each other, it may be necessary to introduce a common time for the system as in the case of Dirac, Fock and Podolsky's theory [6]. The common time may be the proper time τ, or may be the proper time of a proton, for example, being somewhere in space-time. However, an important question now presents itself. Is it absolutely necessary to assume the existence of such a common time? This is a difficult question to answer. We shall not go any further into the analysis of this problem. In what follows we need

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[6] P. A. M. Dirac, V. A. Fock and B. Podolsky: *Phys. Zeits. Sowj.*, **2** 468 (1932).

only to establish some relations between the proper times. We assume that they are connected with each other in the following way:
for the Hamiltonian $\mathfrak{M}_1$

$$m_1\tau_1 = m_2\tau_2 = \ldots = m_n\tau_n = Z\,, \qquad (\hbar = c = 1), \tag{2.1}$$

and for the Hamiltonian $\mathfrak{M}_2$

$$m_1^2\tau_1 = m_2^2\tau_2 = \ldots = m_n^2\tau_n = Z\,, \tag{2.2}$$

where m_i are the masses of particles and Z is a dimensionless quantity. The expressions (2.1) and (2.2) can be readily generalized to the cases of other Hamiltonians. The relations such as (2.1) and (2.2) come from the following consideration. Let us imagine that the wave equation of the i-th particle is described in terms of its external coordinates x_μ. The wave function of the i-th particle depends on its proper time τ_i. We suppose that it is in a stationary state of mass. Therefore the wave function may have a time factor $\exp[im_i\tau_i]$ for the case of the Hamiltonian $\mathfrak{M}_1$. For the other particles one can assume such proper-time factors having different masses and proper times. The exponents of these exponentials are dimensionless quantities. Thus we may be permitted to take up these quantities so as to establish a relation between the proper times. It is for this reason that the relations (2.1) and (2.2) offer a means of comparing the scales of the proper times with each other. These formulas are valid for the time-like quantities τ_i. According to the theory of relativity, it is also assumed that in a complex plane τ_i runs along a real axis, for example, while the other real variable σ_i which is obtained by replacing τ_i by $-i\sigma_i$, lies on an imaginary axis. This point was discussed more fully in paper I. For the σ_i we assume that the relations given below are valid:

$$m_1\sigma_1 = m_2\sigma_2 = \ldots = m_n\sigma_n = X\,, \tag{2.3}$$

and

$$m_1^2\sigma_1 = m_2^2\sigma_2 = \ldots = m_n^2\sigma_n = X\,, \tag{2.4}$$

where X is a dimensionless quantity in the space-like domain, corresponding to Z for the proper time τ_i.

Now, with these conditions and the forms of Hamiltonians described in the previous section, the fundamental equations can be easily written down. As in the three dimensional case, the canonical conjugate quantities are subject to the conditions

$$[x_\mu, P_\nu] = i\delta_{\mu\nu}\,, \quad [y_\mu, p_\nu] = i\delta_{\mu\nu}\,, \quad [x'_\mu, p'_\nu] = i\delta_{\mu\nu}\,, \quad \ldots\,, \tag{2.5}$$

and the momenta are replaced by differential operators according to

$$P_\mu \to -i\frac{\partial}{\partial x_\mu}, \qquad p_\mu \to -i\frac{\partial}{\partial y_\mu}, \qquad p'_\mu \to -i\frac{\partial}{\partial x'_\mu}, \ldots . \tag{2.6}$$

For the individual particle, the wave equation is of the form

$$i\frac{\partial\psi(x_j,\tau_j)}{\partial\tau_j} = \mathfrak{M}(j)\psi(x_j,\tau_j), \qquad (j=1,2,\ldots,n). \tag{2.7}$$

The wave equation for the n-particle system can be written as

$$i\frac{\partial\Psi(x_1,\ldots,x_n,\tau)}{\partial\tau} = \Big(\sum_j \mathfrak{M}(j)\Big)\Psi(x_1,\ldots,x_n,\tau), \tag{2.8}$$

where we have adopted the proper time of the center of mass of this system as the common time. The summation means to include the interactions between particles as well as the free Hamiltonians.

For the *one-body problem* of a Klein-Gordon particle, to which we shall turn our special attention in the following sections, we shall discuss the wave equation more fully by employing the Hamiltonian $\mathfrak{M}_2$ and the first method for the partition of the physical mass m. Namely, according to (1.33)–(1.38) we assume that

$$\mathfrak{M}_2 = 2(p_\mu'^2 + p_\mu''^2) - V(y) = P_\mu^2 + 4p_\mu^2 - V(y), \tag{2.9}$$

and

$$m_1 = m_2 = \frac{m}{2}, \qquad m = m_0 - \delta m. \tag{2.10}$$

The wave equation in the coordinate space for x'_μ and x''_μ is given by

$$i\frac{\partial\Psi(x',x'',\tau)}{\partial\tau} = \left(-2\frac{\partial^2}{\partial x_\mu'^2} - 2\frac{\partial^2}{\partial x_\mu''^2} - V(y)\right)\Psi(x',x'',\tau), \tag{2.11}$$

where τ is the proper time of the center of mass. One might take another proper time instead of τ, which would give rise to a different time factor. It seems, however, convenient to adopt the τ as the common time in view of the fact that we are now merely considering the *one-body problem*. Then in virtue of the relation (1.35) the proper time factor is separated from the other part as follows:

$$\Psi(x',x'',\tau) = \exp[im^2\tau + i\delta m^2\tau]\varphi(x',x''), \tag{2.12}$$

where $-\delta m^2$ indicates the mass correction arising from the self-potential $-V(y)$. It should be noted that the function $\varphi(x', x'')$ is similar to that in the so-called non-local field theory proposed by YUKAWA [1].

Let us go over into the (x_μ, y_μ)-space. Then (2.11) becomes

$$i\frac{\partial \Psi(x, y, \tau)}{\partial \tau} = \left(-\frac{\partial^2}{\partial x_\mu^2} - 4\frac{\partial^2}{\partial y_\mu^2} - V(y)\right)\Psi(x, y, \tau)\,. \tag{2.13}$$

The behaviour of the single particle involving the effect of its self-energy is obtained by solving (2.13). When we assume the solution of (2.13) in the form

$$\Psi(x, y, \tau) = \Phi(x, \tau)\psi(y, \tau)\,, \tag{2.14}$$

we get the system of equations

$$i\frac{\partial \Phi(x, \tau)}{\partial \tau} = \left(-\frac{\partial^2}{\partial x_\mu^2}\right)\Phi(x, \tau)\,, \tag{2.15}$$

and

$$i\frac{\partial \psi(y, \tau)}{\partial \tau} = \left(-4\frac{\partial^2}{\partial y_\mu^2} - V(y)\right)\psi(y, \tau)\,. \tag{2.16}$$

Eq. (2.15) was investigated by FEYNMAN [7] in the more general case when an external field interacts with the particle. The Eq. (2.15) is separated in τ, so that we find

$$\Phi(x, \tau) = \exp\left[im^2\tau\right]\varphi(x)\,, \tag{2.17}$$

where $\varphi(x)$ satisfies the equation

$$\left(\frac{\partial^2}{\partial x_\mu^2} - m^2\right)\varphi(x) = 0\,. \tag{2.18}$$

In this way, we obtain the Klein-Gordon equation for the external co-ordinates of the particle. We assume that the wave function $\varphi(x)$ is the ordinary field function which forms the principal subject of the present field theory, in the case of the Klein-Gordon particle. Therefore, any manipulation of the conventional field theory can be applied to the $\varphi(x)$.

Now, let us consider the wave equation (2.16). As is easily seen from (1.35) or (2.12), the wave function $\psi(y, \tau)$ is considered to have the form

$$\psi(y, \tau) = \exp\left[i\delta m^2\tau\right]\psi(y)\,. \tag{2.19}$$

[7] R. P. FEYNMAN: *Phys. Rev.*, **80**, 440 (1950), Appendix A.

Therefore, we obtain the wave equation

$$\left(4\frac{\partial^2}{\partial y_\mu^2}+V(y)\right)\psi(y)=\delta m^2\psi(y)\,, \tag{2.20}$$

which leads to the *eigenvalue* of the mass correction of the particle. We remark here that we have to impose a condition on the variables y_μ, and at the same time on the self-potential $-V(y)$ in order to solve (2.20). The condition is that the vector y_μ must be space-like. The vector y_μ in space-time can, in principle, be space-like or time-like. However, we confine ourselves to the case of the space-like vector y_μ. This restriction stems from the condition that the wave function $\psi(y)$ has to be *quadratically integrable* over the range of y_μ. In any case, the first step in the solution of Eq. (2.20) is to transform it to polar co-ordinates $(R,\omega,\theta,\varphi)$ in the y_μ-space. Then the wave function is divided into four parts corresponding to the new variables. The wave functions for the angle ω are quadratically integrable only if the vector y_μ is space-like. Therefore, the time-like y_μ is rejected. Accordingly, it is assumed that the self-potential $-V(y)$ depends on the space-like y_μ. However, in the many-particle problem in which actual particles are considered to interact with each other the vector y_μ need not be only space-like because of the possibility that particles of the system may be in time-like regions with respect to the other particles. This may be the case for the scattering problem.

As to the self-potential $-V(y)$, we shall now investigate the procedure which is defined by the so-called renormalization theory. The idea of the renormalization of the mass of the particle can be realized in the following way. First of all, let us consider the Hamiltonian of the actual particle without the self-potential $-V(y)$. From (1.35) and (2.9) it is

$$\mathfrak{M}_2=P_\mu^2+4p_\mu^2=-m^2\,. \tag{2.21}$$

This can be separated into two equations

$$P_\mu^2=-m^2\,, \tag{2.22}$$

and

$$4p_\mu^2=0\,. \tag{2.23}$$

The second equation implies that there is no internal mass in the case of no potential. The first equation gives us the starting point of the renormalization theory. It should be noted that the mass m appearing on the right-hand side of (2.22) is the *observed mass* which includes a bare mass and the mass correction arising from the self-energy. As is well known, the interaction between

the particle considered and other particles (or fields) creates divergent self-energies. This means that some infinite terms are added to the right-hand side of (2.22). However, according to the renormalization theory [8], the divergent terms have to be cancelled out by the counter terms which have been inserted into original interaction terms in the usual Hamiltonian. Let $-\delta m^2$ be the counter term and $-V(0)$ be the divergent negative self-energy, then the Eq. (2.22) is modified as follows:

$$\left\{\begin{aligned} P_\mu^2 &= -(m^2 - V(0)) - \delta m^2 , \\ &= -m^2 + (V(0) - \delta m^2) , \qquad V(0) > 0. \end{aligned}\right. \tag{2.24}$$

The renormalization theory claims that $-\delta m^2$ should be chosen in such a way that the bracketed term on the right-hand side of (2.24) vanishes. Strictly speaking, since the $-V(0)$ is infinite the counter term $-\delta m^2$ is also infinite. Therefore, it is difficult to put the renormalization theory on a rigorous mathematical basis.

Our treatment of the self-energy part is designed to avoid this trouble in the following way. As was shown in our papers I and II, the self-energy is supposed to be a function of the internal co-ordinates y_μ [9]. The divergences take place in the limit of $y_\mu^2 \to 0$. In other words, it becomes infinite on a *light-cone*. So long as we take the value $y_\mu^2 \neq 0$, we have a regular expression of the self-energy as a function of y_μ^2, so that we call it the *self-potential* and denote it by $-V(y)$. Let us consider the Hamiltonian $\mathfrak{M}_2$ including the self-potential $-V(y)$ and the counter term $-\delta m^2$. Then from (2.21) it becomes

$$\left\{\begin{aligned} \mathfrak{M}_2 = P_\mu^2 + 4p_\mu^2 &= -(m^2 - V(y)) - \delta m^2 , \\ &= -m^2 + (V(y) - \delta m^2) . \end{aligned}\right. \tag{2.25}$$

The term involved in the bracket of (2.25) is considered to be a quantity relating to the *internal* space. Consequently, the equations (2.22) and (2.23) can be written as

$$P_\mu^2 = -m^2 , \tag{2.26}$$

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[8] F. J. Dyson: *Phys. Rev.*, **75**, 486 (1949), see Sect. 4, Eq. (22).

[9] This fact is well known in the case of Dirac-Heisenberg's positron theory. P. A. M. Dirac: *Proc. Camb. Phil. Soc.*, **30**, 150 (1934); W. Heisenberg: *Zeits. f. Phys.*, **90**, 209 (1934); R. Serber: *Phys. Rev.*, **49**, 545 (1936).

and

$$4p_\mu^2 = V(y) - \delta m^2 \,. \tag{2.27}$$

Eq. (2.26) corresponds to the Klein-Gordon equation (2.18), while Eq. (2.27) leads to Eq. (2.20). This is the way how one can get a formal connection between the usual theory of renormalization and the present treatment. However, the argument of this paper is quite different from that of the renormalization theory in that it is absolutely not necessary to cancel the divergence by the counter term. It is a remarkable feature of the present method that the self-energy is not an infinite numerical value any more but a regular function at anywhere in the internal space except for the light-cone, and $-\delta m^2$ is a finite numerical value which is to be determined by the wave equation (2.20). From this point of view one may say that the renormalization theory furnishes a phenomenological and technical method in terms of which divergences disappear automatically from an S matrix. In this way we understand that from now on the only thing necessary for us is *not to cancel out the divergences but to transfer them from the external space to the internal space.*

Up to now we have confined our discussion to the case of the Schrödinger picture. The quantum-mechanical behaviour of the particle is described also by an equation in the Heisenberg picture. For every physical quantity F which does not explicitly depend on a proper time τ, we assume the equation of motion

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = -i[F, \mathfrak{M}] \,, \tag{2.28}$$

where τ stands for one of the $\tau, \tau_1, \tau_2, \ldots, \tau_n$ and $\mathfrak{M}$ is the Hamiltonian operator.

3. – Derivation of the Self-Potentials for Pseudoscalar Mesons.

We shall consider the application of our method to actual problems. We will treat the one-body problem from the new point of view and confine our discussion in the following sections to the stationary states for the mass of the particle involved. We would like to make a few remarks, in a general sense, about the model which we are going to investigate. To begin with, one would think it most suitable to study the case of a Dirac particle interacting with an electromagnetic field or with a pseudoscalar meson field in view of the fact that the forms and coupling constants of these interactions are well known compared to other interactions. However, at least in the second-order approximation, these cases lead to *positive* self-potentials, that is, positively divergent self-energies, which have nothing to do with the problem of mass

quantization. This situation is exactly analogous to the corresponding situation in the hydrogen-like atoms with *repulsive* Coulomb potentials in the ordinary quantum mechanics. Thus we are forced to take up another model. We find it most convenient to study the case of a pseudoscalar meson that has *negative* self-potentials caused by the nucleons in the intermediate states. The negative signs of the self-potentials reflect the general feature of the polarization of vacuum. This example furnishes another possibility of investigating the coupling constants between the nucleon and meson in terms of their masses, independently of the usual method by which the coupling constants are determined from the cross-sections for scattering processes, nuclear forces, the magnetic moment of the nucleon, and so forth.

We shall consider the case of the pseudoscalar meson in interaction with the nucleon through the couplings of pseudoscalar and pseudovector types in the external space. Let $\varkappa$ be the observed mass of the meson, and let the nucleon field be described by a spinor field variable $\psi(x)$. The complete Lagrangian density for the system may be written as

$$L = L_0 + L_1 + L_2 + L_3 , \qquad (\hbar = c = 1), \tag{3.1}$$

$$L_0 = -\tfrac{1}{2}[(\partial_\mu \varphi_\alpha(x))^2 + \varkappa^2 \varphi_\alpha^2(x)] , \qquad (\alpha = 1, 2, 3), \tag{3.2}$$

$$L_1 = -\frac{f}{\varkappa}(\partial_\mu \varphi_\alpha(x))\left(\frac{1}{2i}\right)[\bar{\psi}(x), \mathfrak{T}_\alpha \gamma_5 \gamma_\mu \psi(x)] , \tag{3.3}$$

$$L_2 = -g\varphi_\alpha(x)(\tfrac{1}{2})[\bar{\psi}(x), \mathfrak{T}_\alpha \gamma_5 \psi(x)] , \tag{3.4}$$

$$L_3 = \frac{i}{2}[\bar{\psi}(x)\gamma_\mu \partial_\mu \psi(x) - (\partial_\mu \bar{\psi}(x))\gamma_\mu \psi(x)] - M\bar{\psi}(x)\psi(x) . \tag{3.5}$$

Here M stands for the observed nucleon mass (we ignore the mass difference between the proton and neutron), $\mathfrak{T}_\alpha$ are isotopic spins, and f and g are dimensionless coupling constants. The Dirac matrices satisfy the relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\delta_{\mu\nu} , \qquad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_4 ,$$

$$\gamma_5^2 = -1 , \qquad \gamma_4 = i\gamma_0 , \qquad \bar{\psi} = \psi^\dagger \gamma_0 .$$

The equations of motion for the particles are found to be

$$\left(-i\gamma_\mu \partial_\mu - \frac{f}{\varkappa}\mathfrak{T}_\alpha \gamma_5 \gamma_\mu \partial_\mu \varphi_\alpha(x) - g\mathfrak{T}_\alpha \gamma_5 \varphi_\alpha(x) + M\right)\psi(x) = 0 , \tag{3.6}$$

$$(3.7) \qquad -i\partial_\mu\bar{\psi}(x)\gamma_\mu + i\frac{f}{\varkappa}\mathfrak{T}_\nu\bar{\psi}(x)\gamma_5\gamma_\mu\partial_\mu\tau_\nu(x) + g\mathfrak{T}_\nu\varphi_\nu(x)\bar{\psi}(x)\gamma_5 - M\bar{\psi}(x) = 0 ,$$

$$(3.8) \qquad (-\partial_\mu^2 + \varkappa^2)\varphi_\alpha(x) = \langle j_\alpha(x)\rangle .$$

Here the term $\langle j_\alpha(x)\rangle$ indicates the effect due to the nucleon pairs in the intermediate states, and involves the self-energy of the meson.

Our main task in this section is to derive the self-potential $-V(y)$ for the meson in the second-order approximation in the coupling constants. One may be rather suspicious of making use of power series expansions especially for the case of the meson. The conditions for the validity of the power series expansions have not been checked. However, in conforming correctly to the physical interpretation of the present method, at the moment we may be permitted to invoke the method of the power series expansions.

For simplicity, in what follows we only consider the case of the neutral pseudoscalar meson with the pseudovector coupling. That is, let $\mathfrak{T}_\alpha$ be replaced by $\mathfrak{T}_3$, and let $\varphi_\alpha(x)$ be written simply as $\varphi(x)$ being the same wave function as that defined in (2.18). The applications to the other cases are straightforward.

We now wish to carry out our calculations by means of the proper-time formulation developed by SCHWINGER and FEYNMAN ([10]) independently. This formalism has remarkable advantages. First, the divergent self-energies are expressed in terms of the invariant proper times. Then, the physical meaning of the divergences can be clearly understood in a space-time representation. Second, with these expressions of the self-energies, we can treat the problem of mass-quantization in such a way that we investigate the eigenvalue problem of energy in the ordinary quantum mechanics. Third, the proper time formalism can also be applied to the cases which are discussed by the conventional field theory, such as the Lamb shift, the anomalous magnetic moment of the electron, and the radiative correction to the scattering processes ([10]). Therefore, this formalism is sufficiently general to comprise any of the results gained by present field theory. The main aim of this paper is to emphasize how the proper time formalism can be applied to the problem of mass quantization as well as to the problems mentioned above. In his paper, FEYNMAN ([7]) has suggested the eigenvalue problem of mass in connection with his proper time formulation. However, an attempt at a comprehensive theory along his line has not been published.

We begin with introducing the transformation function along the same lines as those of SCHWINGER and FEYNMAN. First of all, we consider the trans-

([10]) J. SCHWINGER: *Phys. Rev.*, **82**, 664 (1951); R. P. FEYNMAN: *Phys. Rev.*, **80**, 440 (1950); R. KARPLUS and A. KLEIN: *Phys. Rev.*, **85**, 972 (1952).

formation function for the wave function $\Phi(x, \tau_1)$ given by (2.17) where τ_1 is the proper time of the meson. It transforms $\Phi(x', 0)$ in a representation in which x'_μ is diagonal at time $\tau_1 = 0$ into $\Phi(x, \tau_1)$ in a representation in which x_μ is diagonal at time $\tau_1 = \tau_1$, where x'_μ is not the co-ordinate x'_μ introduced in the previous sections but the external co-ordinate for the different values of τ_1. According to Eq. (2.15), $\Phi(x, \tau_1)$ obeys the equation

$$i \frac{\partial \Phi(x, \tau_1)}{\partial \tau_1} = -\left(\frac{\partial^2}{\partial x_\mu^2}\right) \Phi(x, \tau_1) . \tag{3.9}$$

Then the transformation function $(x(\tau_1)|x'(0))$ is defined by

$$\Phi(x, \tau_1) = \int (x(\tau_1)|x'(0)) \Phi(x', 0)(\mathrm{d}x') . \tag{3.10}$$

Since the Hamiltonian for Eq. (3.9) is $\mathfrak{M}_2 = P_\mu^2$, the plane wave solution of $\Phi(x, \tau_1)$ is written as

$$\Phi(x, \tau_1) = \exp[-iP^2\tau_1]\varphi(x) , \qquad \varphi(x) = \exp[i(Px)] , \tag{3.11}$$

where (Px) stand for $(P_\mu x_\mu)$. With this solution, the transformation function is defined by

$$(x(\tau_1)|x'(0)) = \sum_n \varphi_n(x)\varphi_n^*(x') \exp[-iP^2\tau_1] , \tag{3.12}$$

where the summation is taken over all possible P_μ.

Inserting (3.11) into (3.12) and employing the integration in the P_μ space, one obtains

$$(x(\tau_1)|x'(0)) = \frac{1}{(2\pi)^4} \int \exp[i(P(x - x') - iP^2\tau_1](\mathrm{d}P) . \tag{3.13}$$

With the help of the mathematical formula

$$\left\{ \begin{aligned} \int (\mathrm{d}P) \exp[iaP^2 + i(Px)] &= \int (\mathrm{d}P) \exp[iaP^2] \exp\left[-i\frac{x^2}{4a}\right] , \\ &= i\frac{\pi^2}{a|a|} \exp\left[-i\frac{x^2}{4a}\right] , \end{aligned} \right. \tag{3.14}$$

(3.13) becomes

$$(x(\tau_1)|x'(0)) = \frac{-i}{(4\pi)^2} \left(\frac{1}{\tau_1^2}\right) \exp\left[i\frac{(x - x')^2}{4\tau_1}\right] . \tag{3.15}$$

It should be noted that the expression (3.15) can be applicable not only to the case of the Klein-Gordon particle but also to the case of the Dirac particle obeying an iterated Dirac equation without interaction by replacing the τ_1 by the proper time τ_2 of the Dirac particle.

The Green's function for the meson is defined by

$$\Delta_+(x-x') = i\int_0^\infty d\tau_1 \exp[-i\varkappa^2\tau_1](x(\tau_1)|x'(0)) , \tag{3.16}$$

$$= \frac{1}{(4\pi)^2}\int_0^\infty \tau_1^{-2}\, d\tau_1 \exp\left[-i\varkappa^2\tau_1 + \frac{(x-x')^2}{4\tau_1}\right], \tag{3.16'}$$

and $\Delta_+(x-x')$ satisfies the relation

$$(-\Box_x + \varkappa^2)\Delta_+(x-x') = \delta(x-x') . \tag{3.16''}$$

An alternative method of defining the $\Delta_+(x-x')$ in terms of the usual expression of it in the momentum space is

$$\Delta_+(x-x') = \frac{1}{(2\pi)^4}\int (dP) \exp[iP(x-x')](P^2+\varkappa^2)^{-1} , \tag{3.17}$$

$$= \frac{i}{(2\pi)^4}\int (dP) \exp[iP(x-x')]\int_0^\infty d\tau_1 \exp[-i\tau_1(P^2+\varkappa^2)] , \tag{3.17'}$$

$$= \frac{1}{(4\pi)^2}\int_0^\infty \tau_1^{-2}\, d\tau_1 \exp\left[-i\varkappa^2\tau_1 + \frac{(x-x')^2}{4\tau_1}\right], \tag{3.17''}$$

which is the same form as that of (3.16').

In order to define the Green's function $G(x, x')$ for the nucleon interacting with the meson through the pseudovector coupling, we employ the following notations:

$$\pi_\mu = k_\mu + i\frac{f}{\varkappa}\mathfrak{T}_3\gamma_5\varphi_\mu , \qquad \varphi_\mu = \frac{\partial\varphi(x)}{\partial x_\mu} , \tag{3.18}$$

where k_μ is the energy-momentum vector of the nucleon and is replaced by $-i\partial_\mu$ in a representation in which x_μ is diagonal. Then, the definition of $G(x, x')$ is as follows:

$$(\gamma_\mu\pi_\mu + M)G(x, x') = \delta(x-x') , \tag{3.19}$$

or

$$G(x, x') = i\langle(\psi(x)\bar{\psi}(x))_{+}\rangle\varepsilon(x - x') , \tag{3.20}$$

where

$$\varepsilon(x - x') = \begin{cases} +1 & x_0 > x_0' , \\ -1 & x_0 < x_0' . \end{cases} \tag{3.21}$$

With the help of (3.20), from (3.3) and (3.8) one obtains

$$L_1 = -\frac{f}{\varkappa}\varphi_\mu[\operatorname{tr} \mathfrak{T}_3\gamma_5\gamma_\mu G(x, x')]_{x'\to x} , \tag{3.22}$$

and

$$\langle j(x)\rangle = \frac{f}{\varkappa}\frac{\partial}{\partial x_\mu}[\operatorname{tr} \mathfrak{T}_3\gamma_5\gamma_\mu G(x, x')]_{x'\to x} . \tag{3.23}$$

The Green's function $G(x, x')$ is considered a matrix element with respect to the co-ordinates x_μ. Then it follows from (3.19) that

$$(\gamma\pi + M)G = 1 . \tag{3.24}$$

We can express $G(x, x')$ in terms of the proper time τ_2 of the nucleon in the following way,

$$G = (\gamma\pi + M)^{-1} = (-\gamma\pi + M)[M^2 - (\gamma\pi)^2]^{-1} , \tag{3.25}$$

$$= (-\gamma\pi + M)i\int_0^\infty d\tau_2 \exp[-i(M^2 - (\gamma\pi)^2)\tau_2] . \tag{3.26}$$

Then, the action integral for the part of the Lagrangian of the nucleon is

$$W_1 = \int (dx)\bar{L}_1(x) , \tag{3.27}$$

where $\bar{L}_1(x)$ is given by

$$\bar{L}_1(x) = \frac{i}{2}\int_0^\infty d\tau_2\tau_2^{-1} \exp[-iM^2\tau_2 + i(\gamma\pi)^2\tau_2] . \tag{3.28}$$

Actually, if we consider an arbitrary variation of φ_μ, we obtain the relation

$$\delta\bar{L}_1(x) = \frac{i}{2}\,\delta\int_0^\infty \mathrm{d}\tau_2\tau_2^{-1}\exp\left[-iM^2\tau_2 + i(\gamma\pi)^2\tau_2\right], \tag{3.29}$$

$$= -\frac{f}{\varkappa}(\delta\varphi_\mu)[\mathrm{tr}\,\mathfrak{T}_3\gamma_5\gamma_\mu G] = \delta L_1\,,$$

in virtue of (3.22) and (3.26).

This shows that $\bar{L}_1(x)$ is supposed to be effectively the same as $L_1(x)$ given by (3.3). The Lagrangian (3.28) is written in another form by means of the transformation function $U(\tau_2)$ for the nucleon. Let us first consider the case of the Klein-Gordon particle. From the expression (3.13), it is easy to see that the energy-momentum representation $U(\tau_1)$ of $(x(\tau_1)\,|\,x'(0))$ is

$$U(\tau_1) = \exp\left[-i\mathfrak{M}_2\tau_1\right] = \exp\left[-iP^2\tau_1\right]. \tag{3.30}$$

Comparing the Green's function $G(x, x')$ defined by (3.26) with $\Delta_+(x - x')$ given by (3.17′), one finds that the transformation function $U(\tau_2)$ for the nucleon in an energy-momentum representation must be of the form

$$U(\tau_2) = \exp\left[-i\mathfrak{M}\tau_2\right] = \exp\left[i(\gamma\pi)^2\tau_2\right], \tag{3.31}$$

where

$$\mathfrak{M} = -(\gamma\pi)^2\,, \tag{3.32}$$

is the Hamiltonian for the nucleon obeying an iterated Dirac equation. The last term of the exponent of (3.28) corresponds to the exponent of (3.31). Thus we get

$$\bar{L}_1(x) = \frac{i}{2}\int_0^\infty \mathrm{d}\tau_2\tau_2^{-1}\exp\left[-iM^2\tau_2\right]\mathrm{Tr}\;U(\tau_2)\,. \tag{3.33}$$

Here Tr means that we take diagonal summations both for spinor indices and for space-time co-ordinates, x_μ. Our next task is to find the expression of the self-energy of the meson due to the nucleons in intermediate states. The Lagrangian density $\bar{L}_1(x)$ contains the meson field $\varphi(x)$ in the part of $U(\tau_2)$. We have to separate the terms having a factor $\varphi^2(x)$. These terms are added to the Lagrangian of the free meson, that is, (3.2), as mass corrections due to the self-energy.

In order to get these terms, one has to calculate Tr $U(\tau_2)$. We shall carry out the calculation by means of the method of perturbation. The transformation

function $U(\tau_2)$ satisfies the differential equation

$$i\frac{\partial U(\tau_2)}{\partial \tau_2} = \mathfrak{M}\, U(\tau_2)\,, \tag{3.34}$$

where $\mathfrak{M}$ is given by (3.32). In the present case the Hamiltonian $\mathfrak{M}$ divides into two parts

$$\mathfrak{M} = \mathfrak{M}^0 + \mathfrak{M}'\,, \tag{3.35}$$

$$\mathfrak{M}^0 = k_\mu^2\,, \tag{3.36}$$

$$\mathfrak{M}' = -i\frac{f}{\varkappa}\mathfrak{T}_3\gamma_5(\gamma_\nu k_\nu)(\gamma_\mu \varphi_\mu) + i\frac{f}{\varkappa}\mathfrak{T}_3\gamma_5(\gamma_\mu\varphi_\mu)(\gamma_\nu k_\nu) + \left(\frac{f}{\varkappa}\right)^2 \mathfrak{T}_3^2(\gamma_\mu\varphi_\mu)(\gamma_\nu\varphi_\nu)\,. \tag{3.37}$$

Correspondingly, the transformation function $U(\tau_2)$ can be written as

$$U(\tau_2) = U_0(\tau_2)U_1(\tau_2)\,, \tag{3.38}$$

where

$$U_0(\tau_2) = \exp\left[-ik^2\tau_2\right], \tag{3.39}$$

and $U_1(\tau_2)$ satisfies the equation

$$i\frac{\partial U_1(\tau_2)}{\partial \tau_2} = \left(U_0^{-1}(\tau_2)\mathfrak{M}'U_0(\tau_2)\right)U_1(\tau_2)\,. \tag{3.40}$$

On inserting (3.38) into (3.34) and using the initial condition $U_1(0) = 1$, we can solve (3.34) by means of the iteration method. Thus $U(\tau_2)$ is found to be

$$U(\tau_2) = U_0(\tau_2) - i\tau_2\left[\mathfrak{M}'U_0(\tau_2)\right] + \\ + \tfrac{1}{2}(-i\tau_2)^2\int_0^1 \mathrm{d}u_1\left[\mathfrak{M}'U_0((1-u_1)\tau_2)\mathfrak{M}'(u_1\tau_2)\right] + \dots\,. \tag{3.41}$$

This expansion will be used for writing down the terms up to the second-order corrections in the coupling constants. By (3.40), Tr $U(\tau_2)$ may be written in the form

$$\mathrm{Tr}\, U(\tau_2) = \mathrm{Tr}\, U_0(\tau_2) + E_1 + E_2\,, \tag{3.42}$$

$$E_1 = -i\tau_2\,\mathrm{Tr}\left[\mathfrak{M}'U_0(\tau_2)\right], \tag{3.43}$$

$$E_2 = \tfrac{1}{2}(-i\tau_2)^2\int_0^1 \mathrm{d}u_1\,\mathrm{Tr}\left[\mathfrak{M}'U_0((1-u_1)\tau_2)\mathfrak{M}'U_0(u_1\tau_2)\right]. \tag{3.44}$$

The evaluation of E_1 and E_2 will be carried out in the co-ordinates space for the nucleon. In the course of calculation, we shall make use only of the momentum representation of the meson field, because we wish to keep the formalism in space-time description as far as possible.

We start with the expression (3.43). The matrix element of the transformation function $U_0(\tau_2)$ has the form

$$(3.45) \qquad (x(\tau_2)|x'(0)) = (x|U_0(\tau_2)|x') = \frac{-i}{(4\pi)^2}\left(\frac{1}{\tau_2}\right)^2 \exp\left[i\frac{(x-x')^2}{4\tau_2}\right],$$

with the help of (3.15).

From the last term of (3.37) and (3.43), one finds,

$$(3.46) \qquad E_1 = -i\tau_2\left(\frac{f}{\varkappa}\right)^2 \mathrm{Tr}\left[\mathfrak{T}_3^2(\gamma_\mu\varphi_\mu)(\gamma_\nu\varphi_\nu)(x(\tau_2)|x'(0))\right],$$

$$= -\frac{1}{(4\pi)^2}\left(\frac{1}{\tau_2}\right)\left(\frac{f}{\varkappa}\right)^2 \mathrm{Tr}\left\{\mathfrak{T}_3^2(\gamma_\mu\varphi_\mu)(\gamma_\nu\varphi_\nu)\exp\left[i\frac{(x-x')^2}{4\tau_2}\right]\right\}.$$

By making use of the trace evaluations,

$$(3.47) \qquad \mathrm{tr}\,\gamma_\nu^2 = -4\,, \quad (\nu = 1, 2, 3, 4)\,, \qquad \mathrm{tr}\,\mathfrak{T}_3^2 = 2\,,$$

one obtains

$$(3.48) \qquad E_1 = \frac{1}{2\tau_2}\left(\frac{f}{\pi\varkappa}\right)^2 \varphi_\mu^2(x)\,.$$

Next, let us consider E_2. In the representation of co-ordinates space, E_2 is of the form

$$(3.49) \qquad E_2 = \tfrac{1}{2}(-i\tau_2)^2 \int_0^1 \int_0^1 \mathrm{d}u_1\, \mathrm{tr}\left[(x|\mathfrak{M}' U_0((1-u_1)\tau_2)|x')(x'|\mathfrak{M}' U_0(u_1\tau_2)|x)\right](\mathrm{d}x')\,,$$

in virtue of (3.44). Here $\mathfrak{M}'$ includes merely linear terms in the coupling constants. $U_0(u_1\tau_2)$ and $U_0((1-u_1)\tau_2)$ have forms such as (3.45). From (3.37) it follows that the matrix elements in (3.49) are in the forms:

$$(3.50) \qquad (x|\mathfrak{M}' U_0((1-u_1)\tau_2)|x') =$$

$$= i\left(\frac{\mathfrak{T}_3 f}{\varkappa}\right)\left(\frac{1}{4\pi(1-u_1)\tau_2}\right)^2 \exp\left[i\frac{(x-x')^2}{4(1-u_1)\tau_2}\right]\cdot$$

$$\cdot\left\{A_1[\gamma_5(\gamma_\nu \Delta x_\nu)(\gamma_\mu\varphi_\mu) - \gamma_5(\gamma_\mu\varphi_\mu)(\gamma_\nu \Delta x_\nu)] + \gamma_5(\gamma_\nu\partial_\nu)(\gamma_\mu\varphi_\mu)\right\},$$

and

$$(3.51)\quad (x'|\mathfrak{M}'U_0(u_1\tau_2)|x) = \\ = i\left(\frac{\mathfrak{T}_3 f}{\varkappa}\right)\left(\frac{1}{4\pi u_1\tau_2}\right)^2 \exp\left[i\frac{(x-x')^2}{4u_1\tau_2}\right]\cdot \\ \cdot\{A_2[-\gamma_5(\gamma_\varrho \Delta x_\varrho)(\gamma_\sigma\varphi'_\sigma) + \gamma_5(\gamma_\sigma\varphi'_\sigma)(\gamma_\varrho\Delta x_\varrho)] + \gamma_5(\gamma_\varrho\partial'_\varrho)(\gamma_\sigma\varphi'_\sigma)\},$$

where

$$(3.52)\quad \varphi'_\sigma = \frac{\partial\varphi(x')}{\partial x'_\sigma}, \qquad \partial'_\varrho = \frac{\partial}{\partial x'_\varrho},$$

$$(3.53)\quad \Delta x_\mu = x_\mu - x'_\mu,$$

$$(3.54)\quad A_1 = \frac{i}{2(1-u_1)\tau_2}, \qquad A_2 = \frac{i}{2u_1\tau_2}.$$

We now evaluate tr[...] of (3.49). The terms involving the factor $\varphi^2(x)$ stem from the quantities being the zeroth- and second-order in Δx of tr[...]. Let us first consider the terms of $(\Delta x)^2$. From (3.50) and (3.51), one gets

$$(3.55)\quad T_1 = \mathrm{tr}\left[\{\gamma_5(\gamma_\nu\Delta x_\nu)(\gamma_\mu\varphi_\mu) - \gamma_5(\gamma_\mu\varphi_\mu)(\gamma_\nu\Delta x_\nu)\}\cdot\right. \\ \left.\cdot\{-\gamma_5(\gamma_\varrho\Delta x_\varrho)(\gamma_\sigma\varphi'_\sigma) + \gamma_5(\gamma_\sigma\varphi'_\sigma)(\gamma_\varrho\Delta x_\varrho)\}\right].$$

With the help of the relation

$$(3.56)\quad \mathrm{tr}(\gamma_\nu\gamma_\mu\gamma_\varrho\gamma_\sigma) = -4(\delta_{\nu\varrho}\delta_{\sigma\mu} - \delta_{\mu\varrho}\delta_{\nu\sigma} - \delta_{\mu\nu}\delta_{\sigma\varrho}),$$

it is easily seen that

$$(3.57)\quad T_1 = -16(\Delta x)^2\varphi_\mu\varphi'_\mu + 16(\Delta x_\mu\varphi_\mu)(\Delta x_\nu\varphi'_\nu).$$

Therefore, (3.49) gives rise to an integral,

$$(3.58)\quad -\frac{16}{(4\pi)^4}\int_0^1\int\mathrm{d}u_1(\mathrm{d}x')\left(\frac{f}{\varkappa(1-u_1)u_1\tau_2}\right)^2 A_1A_2[(\Delta x)^2(\varphi_\mu\varphi'_\mu) - (\Delta x_\mu\varphi_\mu)(\Delta x_\nu\varphi'_\nu)]\cdot \\ \cdot\exp\left[i\left\{\frac{1}{4(1-u_1)\tau_2} + \frac{1}{4u_1\tau_2}\right\}(\Delta x)^2\right].$$

It will be useful to introduce a new variable v and A_0, defined by

$$(3.59)\quad u_1 = \tfrac{1}{2}(1+v), \qquad 1-u_1 = \tfrac{1}{2}(1-v),$$

and

$$A_0 = -A_1 A_2 = \frac{1}{(1-v^2)\tau_2^2}. \tag{3.60}$$

Then (3.44) becomes

$$E_2 = E_2' + E_2'' + E_2''', \tag{3.61}$$

where

$$E_2' = \frac{\tau_2^2}{2\pi^4}\left(\frac{f}{\varkappa}\right)^2 \int_{-1}^{1} A_0^3 \,\mathrm{d}v \int (\mathrm{d}x')(\Delta x)^2(\varphi_\mu \varphi_\mu') \exp\left[iA_0\tau_2(x-x')^2\right], \tag{3.62}$$

$$E_2'' = \frac{\tau_2^2}{2\pi^4}\left(\frac{f}{\varkappa}\right)^2 \int_{-1}^{1} A_0^3 \,\mathrm{d}v \int (\mathrm{d}x')(\Delta x_\mu \varphi_\mu)(\Delta x_\nu \varphi_\nu') \exp\left[iA_0\tau_2(x-x')^2\right], \tag{3.63}$$

and E_2''' is the term being the zeroth-order in Δx, and it will be calculated later on. The evaluations of E_2' and E_2'' are simplified by working with the Fourier transforms of the meson fields, which are

$$\varphi(x) = \frac{1}{(2\pi)^2}\int (\mathrm{d}q) \exp\left[iqx\right]\varphi(q), \qquad (q \equiv P), \tag{3.64}$$

and

$$\varphi(x') = \frac{1}{(2\pi)^2}\int (\mathrm{d}q') \exp\left[iq'x'\right]\varphi(q'). \tag{3.65}$$

The meson fields satisfy the equations

$$\left(\frac{\partial^2}{\partial x_\mu^2} - \varkappa^2\right)\varphi(x) = 0, \qquad (q_\mu^2 + \varkappa^2)\varphi(q) = 0, \tag{3.66}$$

$$\left(\frac{\partial^2}{\partial x_\mu'^2} - \varkappa^2\right)\varphi(x') = 0, \qquad (q_\mu'^2 + \varkappa^2)\varphi(q') = 0. \tag{3.67}$$

It will be shown from (3.62) that

$$E_2' = -\frac{\tau_2^2}{2\pi^4}\left(\frac{f}{\varkappa}\right)^2\left(\frac{1}{2\pi}\right)^4 \int_{-1}^{1} A_0^3 \,\mathrm{d}v \int (\mathrm{d}x')(\mathrm{d}q)(\mathrm{d}q')(q_\mu q_\mu')\varphi(q)\varphi(q') \cdot \tag{3.68}$$

$$\cdot \exp\left[iqx\right](\Delta x)^2 \exp\left[iA_0\tau_2(\Delta x)^2 + i(q'x')\right].$$

We shift the origin of x'_μ, $x'_\mu \to x'_\mu + x_\mu$:

$$E'_2 = -\frac{\tau_2^2}{2\pi^4}\left(\frac{f}{\varkappa}\right)^2\left(\frac{1}{2\pi}\right)^4\int_{-1}^{1} A_0^3\,\mathrm{d}v\int(\mathrm{d}q)(\mathrm{d}q')(q_\mu q'_\mu)\varphi(q)\varphi(q')\cdot \cdot\exp\left[i(qx)+i(q'x)\right]I', \tag{3.69}$$

where

$$I' = \int(\mathrm{d}x')(x')^2\exp\left[iA_0\tau_2(x')^2 + i(q'x')\right]. \tag{3.70}$$

In order to calculate the last integral, we consider the following integral,

$$I = \int(\mathrm{d}x')\exp\left[iA_0\tau_2(x')^2 + i(q'x')\right]. \tag{3.71}$$

Then, there holds

$$I' = -\frac{\partial^2 I}{\partial q'_\nu\,\partial q'_\nu}. \tag{3.72}$$

With the aid of the formula (3.14), from (3.71) one gets

$$I = \frac{i\pi^2}{(A_0\tau_2)^2}\exp\left[-i\frac{q_\nu'^2}{4A_0\tau_2}\right]. \tag{3.73}$$

By differentiating twice both sides of (3.73) with respect to q'_ν there follows

$$I' = -\pi^2\left\{\frac{1}{2(A_0\tau_2)^3} - \frac{iq_\nu'^2}{4(A_0\tau_2)^4}\right\}\exp\left[-i\frac{q_\nu'^2}{4A_0\tau_2}\right]. \tag{3.74}$$

Combining (3.69) with (3.74), one finds

$$E'_2 = -\frac{1}{4\pi^2}\left(\frac{f}{\varkappa}\right)^2\left(\frac{1}{2\pi}\right)^4\int_{-1}^{1}\mathrm{d}v\int(\mathrm{d}q)(\mathrm{d}q')\left(\frac{1}{\tau_2} - \frac{iq_\nu'^2}{2A_0\tau_2^2}\right)(-q_\mu q'_\mu)\cdot \cdot\exp\left[i(qx)+i(q'x)\right]\exp\left[-i\frac{q_\nu'^2}{4A_0\tau_2^2}\right]\varphi(q)\varphi(q').$$

It is easy to return to co-ordinates space. Then this becomes

$$E'_2 = -\frac{1}{4\pi^2}\left(\frac{f}{\varkappa}\right)^2\int_{-1}^{1}\mathrm{d}v\left\{\frac{1}{\tau_2} + \frac{i\varkappa^2}{2}(1-v^2)\right\}\exp\left[\frac{i\varkappa^2\tau_2}{4}(1-v^2)\right]\varphi_\mu^2(x). \tag{3.75}$$

We remark here that the Lagrangian density $\bar{L}_1(x)$ is the integrand in the integral (3.27) with respect to x_μ, so that we can apply the following relation to $\varphi_\mu^2(x)$ in (3.75),

$$\int \frac{\partial\varphi(x)}{\partial x_\mu}\frac{\partial\varphi(x)}{\partial x_\mu}(\mathrm{d}x) = -\int\varphi(x)\frac{\partial^2\varphi(x)}{\partial x_\mu^2}(\mathrm{d}x) = -\varkappa^2\int\varphi^2(x)(\mathrm{d}x)\,. \tag{3.76}$$

In this way, we have the expression

$$E_2' = \left(\frac{f}{2\pi}\right)^2\int_{-1}^{1}\mathrm{d}v\left\{\frac{1}{\tau_2}+\frac{i\varkappa^2}{2}(1-v^2)\right\}\exp\left[\frac{i\varkappa^2\tau_2}{4}(1-v^2)\right]\varphi^2(x)\,. \tag{3.77}$$

Next, we proceed to evaluate E_2'' almost in the same way. Let us consider the integral I'',

$$I'' = \int(\mathrm{d}x')(x_\mu' x_\nu')\exp\left[iA_0\tau_2(x')^2 + i(q'x')\right]. \tag{3.78}$$

Using (3.71), we find

$$I'' = -\frac{\partial^2 I}{\partial q_\mu'\partial q_\nu'} = -\pi^2\left\{\frac{\delta_{\mu\nu}}{2(A_0\tau_2)^3}-\frac{iq_\mu' q_\nu'}{4(A_0\tau_2)^4}\right\}\exp\left[-i\frac{q_\mu'^2}{4A_0\tau_2}\right]. \tag{3.79}$$

We can rewrite (3.63) as

$$E_2'' = -\frac{\tau_2^2}{2\pi^4}\left(\frac{f}{\varkappa}\right)^2\left(\frac{1}{2\pi}\right)^4\int_{-1}^{1}\mathrm{d}v A_0^3\int(\mathrm{d}q)(\mathrm{d}q')(iq_\mu)(iq_\nu')\varphi(q)\varphi(q')\cdot$$
$$\cdot\int(\mathrm{d}x')\exp\left[i(qx)+i(q'x')\right](\Delta x_\mu)(\Delta x_\nu)\exp\left[iA_0\tau_2(\Delta x)^2\right]. \tag{3.80}$$

Shifting the origin of x_μ', $x_\mu'\to x_\mu'+x_\mu$, and using (3.78) and (3.79) we have for E_2''

$$\left\{\begin{aligned} E_2'' &= \frac{1}{4\pi^2}\left(\frac{f}{\varkappa}\right)^2\left(\frac{1}{2\pi}\right)^4\int_{-1}^{1}\mathrm{d}v\int(\mathrm{d}q)(\mathrm{d}q')(iq_\mu)(iq_\nu')\exp\left[i(qx)+i(q'x)\right]\cdot\\ &\qquad\cdot\left\{\frac{\delta_{\mu\nu}}{\tau_2}+\frac{(iq_\mu')(iq_\nu')}{2A_0\tau_2^2}\right\}\exp\left[-i\frac{q_\mu'^2}{4A_0\tau_2}\right]\varphi(q)\varphi(q')\,,\\ &= \frac{1}{4\pi^2}\left(\frac{f}{\varkappa}\right)^2\left(\frac{1}{2\pi}\right)^4\int_{-1}^{1}\mathrm{d}v\int(\mathrm{d}q)(\mathrm{d}q')\exp\left[i(qx)+i(q'x)\right]\cdot\\ &\qquad\cdot\left\{\frac{(iq_\mu)(iq_\mu')}{\tau_2}+\frac{(iq_\mu)(iq_\mu')}{2}\varkappa^2(1-v^2)\right\}\exp\left[\frac{i\varkappa^2\tau_2}{4}(1-v^2)\right]\varphi(q)\varphi(q')\,. \end{aligned}\right. \tag{3.81}$$

With the aid of (3.76), it is convenient to write

$$E_2'' = -\left(\frac{f}{2\pi}\right)^2 \int_{-1}^{1} \mathrm{d}v \left\{\frac{1}{\tau_2} + \frac{i\varkappa^2}{2}(1-v^2)\right\} \exp\left[\frac{i\varkappa^2\tau_2}{4}(1-v^2)\right] q^2(x) \,. \tag{3.82}$$

From (3.77) and (3.82), one finds that

$$E_2' + E_2'' = 0 \,. \tag{3.83}$$

The next step is the calculation of E_2''' arising from the zeroth-order terms in Δx. According to (3.50) and (3.51), the trace of this part can be computed directly,

$$\begin{aligned} T_2 &= \mathrm{tr}\,[\gamma_5(\gamma_\nu\partial_\nu)(\gamma_\mu\varphi_\mu)\gamma_5(\gamma_\varrho\partial_\varrho')(\gamma_\sigma\varphi_\sigma')] \\ &= 4[(\partial_\nu\varphi_\mu)(\partial_\nu'\varphi_\mu') - (\partial_\nu\varphi_\mu)(\partial_\mu'\varphi_\nu') - (\partial_\mu\varphi_\mu)(\partial_\nu'\varphi_\nu')] \,, \end{aligned} \tag{3.84}$$

owing to the formula (3.56). The expression of E_2''', by (3.44), (3.50) and (3.51), becomes

$$\begin{aligned} E_2''' = -\frac{1}{8\pi^4}\left(\frac{f}{\varkappa}\right)^2 \int_{-1}^{1} \mathrm{d}v \int (\mathrm{d}x')\, A_0^2\tau_2^2 \,[&(\partial_\nu\varphi_\mu)(\partial_\nu'\varphi_\mu') - (\partial_\nu\varphi_\mu)(\partial_\mu'\varphi_\nu') - \\ &- (\partial_\mu\varphi_\mu)(\partial_\nu'\varphi_\nu')] \exp\left[iA_0\tau_2(x-x')^2\right] \,. \end{aligned} \tag{3.85}$$

When we make use of the Fourier representations of the meson fields, and of the relations (3.71) and (3.73), we notice that

$$\begin{cases} (\partial_\nu\varphi_\mu)(\partial_\nu'\varphi_\mu') - (\partial_\nu\varphi_\mu)(\partial_\mu'\varphi_\nu') \to 0 \,, \\ -(\partial_\mu\varphi_\mu)(\partial_\nu'\varphi_\nu') \to -\varkappa^2\varphi^2 \,. \end{cases} \tag{3.86}$$

Therefore, finally we obtain

$$E_2''' = -\frac{i}{8}\left(\frac{f\varkappa}{\pi}\right)^2 \int_{-1}^{1} \mathrm{d}v\, \varphi^2(x) \exp\left[\frac{i\varkappa^2\tau_2}{4}(1-v^2)\right] \,. \tag{3.87}$$

It is now possible to write the second-order mass corrections in the Lagrangian density $\bar{L}_1(x)$ given by (3.33) in the detailed form

$$\begin{aligned} \bar{L}_1(x) &= \frac{i}{2}\int_0^\infty \mathrm{d}\tau_2\tau_2^{-1} \exp\left[-iM^2\tau_2\right](E_1 + E_2' + E_2'' + E_2''') = \\ &= \frac{-i}{4\pi^2} f^2\varphi^2(x) \int_0^\infty \mathrm{d}\tau_2\tau_2^{-2} \exp\left[-iM^2\tau_2\right] + \left(\frac{1}{4\pi}\right)^2 f^2\varkappa^2 q^2(x) \int_0^\infty \mathrm{d}\tau_2\tau_2^{-1} \exp\left[-iM^2\tau_2\right] \cdot \\ &\quad \cdot \int_{-1}^{1} \mathrm{d}v \exp\left[\frac{i\varkappa^2\tau_2}{4}(1 \quad v^2)\right] \,. \end{aligned} \tag{3.88}$$

Here we made use of (3.76) and ignore the terms independent of the coupling constant f^2. The integration formula

$$\int_{-1}^{1} dv \exp\left[\frac{i\varkappa^2\tau_2}{4}(1-v^2)\right] = 2 + \frac{i\varkappa^2\tau_2}{2}\int_{-1}^{1} dv\, v^2 \exp\left[\frac{i\varkappa^2\tau_2}{4}(1-v^2)\right], \tag{3.89}$$

then transforms (3.88) into

$$\begin{aligned}\bar{L}_1(x) = {} & \frac{-i}{4\pi^2} f^2\varphi^2(x)\int_0^\infty d\tau_2\tau_2^{-2}\exp\left[-iM^2\tau_2\right] + \\ & + \frac{1}{8\pi^2} f^2\varkappa^2\varphi^2(x)\int_0^\infty d\tau_2\tau_2^{-1}\exp\left[-iM^2\tau_2\right] + \\ & + \frac{i}{2(4\pi)^2} f^2\varkappa^4\varphi^2(x)\int_0^\infty\int_{-1}^{1} d\tau_2\, dv\, v^2\exp\left[-iM^2\tau_2 + \frac{i\varkappa^2\tau_2}{4}(1-v^2)\right].\end{aligned} \tag{3.90}$$

The first term on the right-hand side is a *linearly* divergent τ_2-integral. The second term shows that the divergence is only *logarithmic*. The third term is a constant term. We emphasize that the highest singularity of the divergences for the present case is linear. The situation is different when one employs the usual method of calculation. That is to say, instead of utilizing the proper time τ_2, one might have used the more familiar variables k_μ and the expression for the Green's function such as (3.17). This is the usual method of integrations in the k space, which are carried out according to FEYNMAN's procedure, for example. One would be left with *quadratically* divergent integrals for this case. Therefore, in our case the integral reduces from a quadratically to a linearly divergent one [11].

Now, we shall briefly mention some results obtained in the case of the pseudoscalar coupling. The expressions corresponding to those for the $PS(Ps)$ coupling are shown below.

$$\mathfrak{M}'(ps) = -g\gamma_5(\gamma_\mu k_\mu)\varphi + g\gamma_5\varphi(\gamma_\nu k_\nu) + g^2\varphi^2\,, \tag{3.37'}$$

$$E_1(ps) = -i\tau_2\,\mathrm{Tr}\left[\mathfrak{M}'(ps)U_0(\tau_2)\right] = -\frac{1}{2\pi^2}\left(\frac{1}{\tau_2}\right)g^2\varphi^2\,, \tag{3.48'}$$

$$\begin{aligned}T_1(ps) = \mathrm{tr}\,\big[&\{\gamma_5(\gamma_\mu\Delta x_\mu)\varphi - \gamma_5\varphi(\gamma_\nu\Delta x_\nu)\}\cdot \\ &\cdot\{-\gamma_5(\gamma_\varrho\Delta x_\varrho)\varphi' + \gamma_5\varphi'(\gamma_\sigma\Delta x_\sigma)\}\big] = 0\,,\end{aligned} \tag{3.55'}$$

[11] B. J. MALENKA: *Phys. Rev.*, **85**, 686 (1952).

$$T_2(ps) = \mathrm{tr}\,[\gamma_5(\gamma_\mu\partial_\mu)\varphi\gamma_5(\gamma_\varrho\partial'_\varrho)\varphi'] = -4(\partial_\mu\varphi\,\partial'_\mu\varphi')\,, \tag{3.84'}$$

$$E_2'''(ps) = \frac{1}{8\pi^1}\,g^2\int_{-1}^{1}\mathrm{d}r\int(\mathrm{d}x')A_0^2\tau_2^2(\partial_\mu\varphi(x))(\partial'_\mu\varphi(x'))\,\exp[iA_0\tau_2(x\quad x')^2] \tag{3.85'}$$

$$= -\frac{i}{8}\left(\frac{g\varkappa}{\pi}\right)^2\int_{-1}^{1}\mathrm{d}v\,\varphi^2(x)\exp\left[\frac{i\varkappa^2\tau_2}{4}\,(1-v^2)\right]. \tag{3.87'}$$

Therefore, the effective Lagrangian is identical with that of the pseudovector coupling, only if g is replaced by f, at least in the second-order approximation. The explicit evaluation of the self-potential is thereby accomplished.

We now proceed to investigate the connection between the proper times and the classical interval in space-time. This will be done in the Heisenberg picture. It is not difficult to see that in the approximation considered the interaction terms appearing in the space-time intervals contribute to higher order terms. Therefore, such terms may be omitted in the following discussion.

Suppose that the Hamiltonian $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are defined by

$$\mathfrak{M}_1 = \frac{k_\mu^2}{M}\,, \tag{3.91}$$

and

$$\mathfrak{M}_2 = k_\mu^2\,, \tag{3.92}$$

respectively. First, let us consider the case of $\mathfrak{M}_1$. From the equation of motion (2.28), it follows that, for the operators k_μ and x_μ,

$$\frac{\mathrm{d}k_\mu}{\mathrm{d}\tau_2} = -i[k_\mu, \mathfrak{M}_1] = 0\,, \tag{3.93}$$

and

$$\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau_2} = -i[x_\mu, \mathfrak{M}_1] = \frac{2k_\mu}{M}\,, \tag{3.94}$$

where use is made of the relation

$$[x_\mu, k_\nu] = i\delta_{\mu\nu}\,. \tag{3.95}$$

After the integrations over τ_2, we find

$$k_\mu(\tau_2) = k_\mu(0)\,, \tag{3.96}$$

and

$$(x_\mu(\tau_2) - x_\mu(0))\left(\frac{1}{\tau_2}\right) = 2\left(\frac{k_\mu(0)}{M}\right). \tag{3.97}$$

Consequently, ignoring the term arising from

$$[x_\mu(\tau_2), x_\nu(0)] = \left[x_\mu(0) + 2\tau_2\left(\frac{k_\mu(0)}{M}\right), x_\nu(0)\right] = -2i\left(\frac{\tau_2}{M}\right)\delta_{\mu\nu},$$

one obtains

$$(x_\mu(\tau_2) - x_\mu(0)^2) \cong 4\tau_2^2\left(\frac{k_\mu}{M}\right)^2 = -4\tau_2^2. \tag{3.98}$$

In a similar way, for the Hamiltonian (3.92) one gets

$$(x_\mu(\tau_2) - x_\mu(0))^2 \cong -4M^2\tau_2^2. \tag{3.99}$$

If we consider the other form of the Hamiltonian such as

$$\mathfrak{M}_3 = \frac{k_\mu^2}{2M}, \tag{3.100}$$

we find

$$(x_\mu(\tau_2) - x_\mu(0))^2 \cong -\tau_2^2, \tag{3.101}$$

which is the same relation as that in the theory of relativity, and indicates the time-like character of τ_2. When the path of the integration is changed from the τ_2 axis to the σ_2 axis in the complex (τ_2, σ_2)-plane, one finds from (3.98), (3.99) and (3.101) that

$$(x_\mu(\sigma_2) - x_\mu(0))^2 \cong 4\sigma_2^2, \tag{3.102}$$

$$(x_\mu(\sigma_2) - x_\mu(0))^2 \cong 4M^2\sigma_2^2, \tag{3.103}$$

and

$$(x_\mu(\sigma_2) - x_\mu(0))^2 \cong \sigma_2^2. \tag{3.104}$$

These relations will be used in discussing the self-potentials. We call attention to the fact that the expressions (3.102)–(3.104) are applicable not only to the nucleon but also to the mesons, by changing the notations from M and σ_2 to $\varkappa$ and σ_1 respectively.

We now return to the Lagrangian (3.90). Let us first consider the linearly divergent part of it. In order to show the divergence clearly, we replace the

lower limit, O, of the τ_2 integration by a cut-off factor, which is not a constant numerical value but a *variable*. This procedure was explained more fully in the paper I. Then the first integral of (3.90) becomes

$$\overline{L}_{1.\,\mathrm{lin.}}(x) = \frac{-i}{4\pi^2} f^2\varphi^2(x) \int_{i\tau_2}^{\infty} \mathrm{d}\tau_2 \tau_2^{-2} \exp\left[-iM^2\tau_2\right].$$

This can be transformed into the following form as a result of the transformation in the (τ_2, σ_2)-plane:

$$\overline{L}_{1.\,\mathrm{lin.}}(x) = \frac{1}{4\pi^2} f^2\varphi^2(x) \int_{\sigma_2}^{\infty} \mathrm{d}\sigma_2 \sigma_2^{-2} \exp\left[-M^2\sigma_2\right]. \tag{3.105}$$

This shows that in view of the relation (3.103), when we approach the light-cone, whose vertex is at the point $x_\mu(0)$, in such a way that we always restrict ourselves to the region outside the light-cone, we get a positive infinity just on the light-cone.

The mathematical expression

$$\int_{\sigma_2}^{\infty} \frac{\exp\left[-M^2\sigma_2\right]}{\sigma_2^2} \mathrm{d}\sigma_2 = \frac{\exp\left[-M^2\sigma_2\right]}{\sigma_2} - M^2 \int_{\sigma_2}^{\infty} \frac{\exp\left[-M^2\sigma_2\right]}{\sigma_2} \mathrm{d}\sigma_2 , \tag{3.106}$$

permits Eq. (3.105) to be written

$$\overline{L}_{1.\,\mathrm{lin.}}(x) = L_{1.1} + L_{1.2} , \tag{3.107}$$

$$L_{1.1} = \frac{1}{4\pi^2} f^2\varphi^2(x) \frac{\exp\left[-M^2\sigma_2\right]}{\sigma_2} , \tag{3.108}$$

$$L_{1.2} = -\frac{1}{4\pi^2} f^2 M^2 \varphi^2(x) \int_{\sigma_2}^{\infty} \frac{\exp\left[-M^2\sigma_2\right]}{\sigma_2} \mathrm{d}\sigma_2 . \tag{3.109}$$

The last integral shows a logarithmic divergence in the limit of $\sigma_2 \to 0$.

In the same way, the other terms of the Lagrangian (3.90) can be expressed as follows. Namely, the second term of (3.90) is written as

$$\overline{L}_{1.\,\log.}(x) = \frac{1}{8\pi^2} f^2\varkappa^2\varphi^2(x) \int_{\sigma_2}^{\infty} \frac{\exp\left[-M^2\sigma_2\right]}{\sigma_2} \mathrm{d}\sigma_2 . \tag{3.110}$$

The last term of (3.90) leads to

$$\bar{L}_{1\ \mathrm{fin.}}(x) = \frac{1}{2(4\pi)^2} f^2\varkappa^4\varphi^2(x) \int_{\sigma_2}^{\infty} \int_{-1}^{1} \mathrm{d}\sigma_2\, \mathrm{d}v\, v^2 \exp\left[- M^2\sigma_2 + \frac{\varkappa^2\sigma_2}{4}(1 - v^2)\right].$$

In the limit of $\sigma_2 \to 0$, one gets a finite result:

$$\bar{L}_{1.\ \mathrm{fin.}}(x) = \frac{1}{8\pi^2} f^2\varkappa^2\varphi^2(x)\left[2 - \sqrt{p}\ \mathrm{tg}^{-1}\left(\frac{2\sqrt{p}}{p-1}\right)\right],$$

where

$$p = 4\left(\frac{M}{\varkappa}\right)^2 - 1\,.$$

For the π-meson, one finds

$$\left[2 - \sqrt{p}\ \mathrm{tg}^{-1}\left(\frac{2\sqrt{p}}{p-1}\right)\right] = 0.0037\,.$$

This is a very small numerical value. This term will be negligible compared to the other terms.

According to (3.2), the definition of the mass term in the free Lagrangian is $-\frac{1}{2}\varkappa^2\varphi^2(x)$. Therefore, the additional masses arising from the interactions are treated in such a way that the self-potential U appears as

$$-\tfrac{1}{2}\delta\varkappa^2\varphi^2(x) \to -\tfrac{1}{2}U\varphi^2(x)\,,$$

n the Lagrangian. Thus using (3.108), (3.109) and (3.110), we find

(3.111) $$U = U' + U''\,,$$

i

(3.112) $$U' = -\frac{1}{2\pi^2} f^2 \frac{\exp[-M^2\sigma_2]}{\sigma_2}\,,$$

(3.113) $$U'' = \frac{1}{2\pi^2} f^2 M^2 \left\{1 - \frac{1}{2}\left(\frac{\varkappa}{M}\right)^2\right\} \int_{\sigma_2}^{\infty} \frac{\exp[-M^2\sigma_2]}{\sigma_2}\,\mathrm{d}\sigma_2\,.$$

In order to express U in terms of the absolute distance X introduced in (2.4), it is necessary to consider the proper time factor of the wave function of the nucleon. From (3.31) and (3.36), it is easily seen that it is of the form $\exp[iM^2\tau_2]$, so that X is given by

(3.114) $$M^2\sigma_2 = X\,.$$

When we apply (3.114) to (3.112) and (3.113), the self-potentials reduce finally to the forms:

$$U'(X) = -V_1(X) = -\frac{1}{2\pi^2} f^2 M^2 \frac{\exp[-X]}{X}, \tag{3.115}$$

$$U''(X) = -V_2(X) = -\frac{1}{2\pi^2} f^2 M^2 \left\{1 - \frac{1}{2}\left(\frac{\varkappa}{M}\right)^2\right\} E_i(-X), \tag{3.116}$$

where

$$-E_i(-X) = \int_X^\infty \frac{\exp[-X]}{X} \mathrm{d}X > 0 .$$

Then the self-potential defined by

$$V(X) = V_1(X) + V_2(X), \tag{3.117}$$

behaves in the following way. For all $X > 0$, $V(X)$ is positive, for $X \to \infty$, $V(X)$ becomes zero, and for small X, $V(X) \sim V_1(X)$, because in this region $V_1(X)$ is predominant compared to $V_2(X)$ which is always negative if $\varkappa < M$. Thus we have essentially a Yukawa type self-potential for small X, and in addition to it we get the effect of the screening by $V_2(X)$ for large X.

As a final remark of this section we consider the relation between X and the internal coordinates y_μ which were introduced in the previous section. We recall the partition of the actual mass $\varkappa$ into equal parts, which was discussed in Sect. 1. Namely, the center of mass x_μ has the actual mass $\varkappa$, while the fictitious points x'_μ and x''_μ are assumed to have the masses, $\varkappa/2$ and $\varkappa/2$ respectively. Under these conditions, let us turn our attention to the mass point x'_μ and its Hamiltonian given by (1.33). It is written as

$$\mathfrak{M}' = 2p'^2_\mu = -\frac{\varkappa^2}{2}, \tag{3.118}$$

where p'_μ satisfies

$$[x'_\nu, p'_\mu] = i\delta_{\mu\nu} . \tag{3.119}$$

As in the case of (3.102)–(3.104), using the equation of motion (2.28) and the Hamiltonian (3.118), we find

$$(x'_\mu(\sigma') - x'_\mu(0))^2 \cong 4\varkappa^2(\sigma')^2 . \tag{3.120}$$

We can understand the result (3.120) in the following way. We may imagine that the mass point x'_μ travels in space-time as if it were a free particle having

the mass $\varkappa/2$. The world line of this particle is defined by $x'_\mu(\tau')$, τ' being the proper time of the mass point x'_μ. The relation (3.120) shows that, when we consider a point $x'_\mu(0)$ on the world line at a time $\tau' = 0$, the distance between $x'_\mu(0)$ and a point $x'_\mu(\sigma')$ situated in a space-like region with respect to the point $x'_\mu(0)$, is approximately equal to $2\varkappa\sigma'$. Suppose that the second mass point x''_μ is at the point $x'_\mu(\sigma')$, then there follows

$$y_\mu^2 = (x'_\mu - x''_\mu)^2 = (x'_\mu(\sigma') - x'(0))^2 \cong 4\varkappa^2(\sigma')^2 . \tag{3.121}$$

This can be possible since the internal coordinate vector y_μ is assumed to be space-like. Now the relation between X and σ' stems from the Hamiltonian (3.118):

$$\frac{\varkappa^2}{2}\sigma' = X . \tag{3.122}$$

Finally, with the aid of (3.121) and (3.122), one gets

$$X = \frac{\varkappa}{4}\sqrt{y_\mu^2} , \qquad (y_\mu^2 \geqslant 0), \tag{3.123}$$

at least in the present approximation. From (3.117) and (3.123) it is permitted to consider that

$$V(X) = V(y) . \tag{3.124}$$

We assume that $V(y)$ is the self-potential which corresponds to that discussed in the previous sections. One can in this way write down the self-potentials in terms of the internal coordinates so long as one postulates the fundamental relations such as (2.3) and (2.4). However, there is no proof of the uniqueness of such relations. It is required by the analogy with the many-time theory of Dirac, Fock and Podolsky. In spite of the arbitrariness, the problem of mass-quantization will be treated in the succeeding sections in order to illustrate how the use of the self-potential or divergent self-energy, which are supposed to be *meaningless* physical quantities from the conventional point of view, as a *meaningful* physical observable in the generalized wave equation, leads to some quantized mass values.

4. – Evaluation of Mass Eigenvalues in Terms of Power Series Expansions.

In this section we shall consider the following problem. Suppose that the meson has only the self-potential $V_1(X)$ given by (3.115), which is predominant for the small values of X. Then, it may be possible to obtain an ap-

proximate mass spectrum from the wave equation (2.20) by using $V_1(X)$. The effect of $V_2(X)$ defined by (3.116) will be taken into account in the next section. The self-potential $V_1(X)$ is of the so-called Yukawa potential type in the deuteron problem. In such a case, it is not so easy to solve the wave equation and to get the expression of the mass eigenvalues in an analytical form. In the following, an account will, however, be given of methods of obtaining the wave functions and of approximating to the mass eigenvalues by means of the method of inverse power-series expansions with respect to the coupling constants. Therefore, the mass eigenvalues will be given as a function of the coupling constants, quantum numbers and other parameters.

Now the basic equation from which we start is given by (2.20) by replacing m by $\varkappa$:

$$\frac{\partial^2\psi(y)}{\partial y_\mu^2} = -\tfrac{1}{4}V(y)\psi(y) + \tfrac{1}{4}\delta\varkappa^2\psi(y)\,, \tag{4.1}$$

where $V(y)$ is the self-potential defined by (3.124) and $\delta\varkappa^2$ is the numerical mass correction to be determined by this wave equation.

In order to solve Eq. (4.1), it is convenient to transform it to polar co-ordinates in the y_μ space. Since y_μ is a space-like vector, the polar co-ordinates are defined by the following relations ([12]):

$$\begin{cases} y_1 = R\cosh\omega\sin\theta\cos\varphi\,, & 0 < R < \infty, \\ y_2 = R\cosh\omega\sin\theta\sin\varphi\,, & -\infty < \omega < \infty, \\ y_3 = R\cosh\omega\cos\theta\,, & 0 \leqslant \theta < \pi, \\ y_0 = R\sinh\omega\,, & 0 \leqslant \varphi < 2\pi. \end{cases} \tag{4.2}$$

The volume element is

$$\mathrm{d}V = \mathrm{d}y_0\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 = R^3\cosh^2\omega\sin\theta\,\mathrm{d}R\,\mathrm{d}\omega\,\mathrm{d}\theta\,\mathrm{d}\varphi\,. \tag{4.3}$$

Then it is easily proved that there holds

$$\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+\frac{\partial^2}{\partial y_3^2}-\frac{\partial^2}{\partial y_0^2} = \frac{\partial^2}{\partial R^2}+\frac{3}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial\omega^2}-\frac{2\,\mathrm{tgh}\,\omega}{R^2}\frac{\partial}{\partial\omega}+ \tag{4.4}$$

$$+\frac{1}{R^2\cosh^2\omega}\left[\frac{\partial^2}{\partial\theta^2}+\mathrm{ctg}\,\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right].$$

([12]) M. Born and K. Fuchs: *Proc. Roy. Soc. Edinb.*, **60**, 100, 141 (1940); K. Fuchs: *Proc. Roy. Soc. Edinb.*, **60**, 147 (1940).

By setting

$$\psi(y) = \varphi(R)W(\omega)Y_{lm}(\theta, \varphi)\,, \tag{4.5}$$

and introducing the operators

$$M_0 = -\left(\frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\varphi^2}\right), \tag{4.6}$$

$$K^2 = \frac{\partial^2}{\partial\omega^2} + 2\operatorname{tgh}\omega\,\frac{\partial}{\partial\omega} + \frac{M_0}{\cosh^2\omega}\,, \tag{4.7}$$

it can be verified that $W(\omega)$, and $Y_{lm}(\theta, \varphi)$ satisfy the following equations

$$M_0 Y_{lm}(\theta, \varphi) = l(l+1)Y_{lm}(\theta, \varphi)\,, \quad l = 0, 1, 2, \ldots, \ |m| \leqslant l, \tag{4.8}$$

and

$$K^2 W(\omega) = \lambda W(\omega)\,, \qquad (\lambda\text{: eigenvalue}). \tag{4.9}$$

The last equation is written as

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\omega^2} + 2\operatorname{tgh}\omega\,\frac{\mathrm{d}W}{\mathrm{d}\omega} + \left\{\frac{l(l+1)}{\cosh^2\omega} - \lambda\right\} W = 0\,. \tag{4.10}$$

We first substitute in (4.10)

$$W = B(\cosh\omega)^{-1}\,, \tag{4.11}$$

we then get

$$\frac{\mathrm{d}^2 B}{\mathrm{d}\omega^2} + \left\{\frac{l(l+1)}{\cosh^2\omega} + \lambda'\right\} B = 0\,, \tag{4.12}$$

where

$$\lambda' = -(\lambda + 1)\,. \tag{4.13}$$

The functions W and B are to be normalized:

$$\int_{-\infty}^{\infty} W^* W \cosh^2\omega\,\mathrm{d}\omega = \int_{-\infty}^{\infty} B^* B\,\mathrm{d}\omega = 1\,. \tag{4.14}$$

The differential equation (4.12) is easily solved by means of the factorization

method discussed by INFELD and HULL ([13]). The solutions are given by

$$B_q^q(\omega) = \left[\frac{2^{1-2q}\Gamma(2q)}{\Gamma(q)\Gamma(q)}\right]^{\frac{1}{2}} \cosh^{-q}\omega\,, \qquad q > 0, \tag{4.15}$$

$$B_l^q(\omega) = [(l-q)(l+q)]^{\frac{1}{2}}\left(l\,\mathrm{tgh}\,\omega - \frac{\mathrm{d}}{\mathrm{d}\omega}\right) B_{l-1}^q(\omega)\,. \tag{4.16}$$

The corresponding eigenvalues are

$$\lambda' = -q^2\,, \quad \text{i.e.} \quad \lambda = (q-1)(q+1)\,, \tag{4.17}$$

$$l - q = 0, 1, 2, \ldots\,, \qquad q \leqslant l. \tag{4.18}$$

It should be remarked that the degeneracy of q is not finite. In general, there are infinite many values of l corresponding to a given value of q. This is shown in the relation (4.18). Thus the state corresponding to a given value of q has an infinite degeneracy ([14]). This is a characteristic feature of the angular momentum in a Lorentz space: y_μ is supposed to be Lorentz-invariant with respect to homogeneous and inhomogeneous Lorentz transformations. However, up to now the only Lorentz-invariant quantum number is q. We cannot specify the substates for the quantum number l and m without destroying the Lorentz-invariance. The solutions for $q < 0$ are not independent of those for $q > 0$. In the case of $q = 0$, we have to exclude the solution according to the following reason. Transforming to the new variable

$$z = \mathrm{tgh}\,\omega\,, \tag{4.19}$$

one can reduce the condition (4.14) to the form

$$\int_{-\infty}^{\infty} B_q^{q*} B_q^q \,\mathrm{d}\omega = \int_{-1}^{1} (1-z^2)^{q-1}\,\mathrm{d}z\,. \tag{4.20}$$

For $q = 0$, one finds

$$\int_{-1}^{1} (1-z^2)^{-1}\,\mathrm{d}z = \frac{1}{2}\left[\log\frac{1+z}{1-z}\right]_{-1}^{1} \to +\infty\,. \tag{4.21}$$

Therefore, the solutions for $q = 0$ cannot satisfy the quadratically integrable condition.

([13]) L. INFELD and T. E. HULL: *Rev. Mod. Phys.*, **23**, 21 (1951).
([14]) E. SCHRÖDINGER: *Proc. Roy. Irish Acad.*, **55**, 29 (1952).

The operator K^2 is connected with the angular momenta in the y_μ space. In order to illustrate the relation, we define the angular momentum $L_{\mu\nu}$ of the meson in the y_μ space by

$$L_{\mu\nu} = y_\mu p_\nu - y_\nu p_\mu \,. \tag{4.22}$$

Then it follows that

$$K^2 = \tfrac{1}{2}(L_{\mu\nu}L_{\mu\nu}) \,. \tag{4.23}$$

Introducing the notations,

$$M_a = L_{bc} \,, \qquad (a, b, c = 1, 2, 3 \text{ cyclic}) \tag{4.24}$$

$$N_a = L_{4a} \,, \tag{4.25}$$

$$M^2 = (M_a M_a) \,, \tag{4.26}$$

$$N^2 = (N_a N_a) \,, \tag{4.27}$$

one can show that

$$[M_a, M_b] = [N_a, N_b] = iM_c \,, \tag{4.28}$$

$$[M_a, N_b] = [N_a, M_b] = iN_c \,, \tag{4.29}$$

$$[N_a, N_a] = [M_a, M_a] = 0 \,, \tag{4.30}$$

and

$$[K^2, M_a] = [K^2, N_a] = 0 \,. \tag{4.31}$$

From (4.1), (4.4) and (4.17), it is seen that the radial wave function $\varphi(R)$ satisfies the equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}R^2} + \frac{3}{R}\frac{\mathrm{d}}{\mathrm{d}R} - \frac{(q-1)(q+1)}{R^2} + \tfrac{1}{4}(V(R) - \delta\varkappa^2)\right]\varphi(R) = 0 \,, \tag{4.32}$$

$$q = 1 \,,\ 2 \,,\ 3 \,,\ \ldots \,.$$

The wave function is normalized as follows:

$$\int_0^\infty \varphi^*(R)\varphi(R)R^3 \,\mathrm{d}R = 1 \,. \tag{4.33}$$

Writing

$$\varphi(R) = R^{-\frac{3}{2}} u(R) , \tag{4.34}$$

we obtain the equation

$$\frac{\mathrm{d}^2 u(R)}{\mathrm{d}R^2} - \left[\tfrac{1}{4}(\delta\varkappa^2 - V(R)) + \frac{(q-1)(q+1) + \frac{3}{4}}{R^2} \right] u(R) = 0 . \tag{4.35}$$

It is convenient to define the new quantum number

$$Q = q - \tfrac{1}{2} , \qquad Q = \tfrac{1}{2} , \ \tfrac{3}{2} , \ \tfrac{5}{2} , \ \ldots , \tag{4.36}$$

and to change the independent variable R into the absolute distance X according to the relation (3.123):

$$\frac{\varkappa}{4}\sqrt{y_\mu^2} = \frac{\varkappa}{4} R = X . \tag{4.37}$$

The equation then reduces to

$$\frac{\mathrm{d}^2 u(X)}{\mathrm{d}X^2} - \left[(\delta\varkappa^2 - V(X)) \left(\frac{4}{\varkappa^2}\right) + \frac{Q(Q+1)}{X^2} \right] u(X) = 0 . \tag{4.38}$$

In virtue of (3.115) we assume as a first approximation

$$V(X) \cong V_1(X) = \frac{1}{2\pi^2} f^2 M^2 \frac{\exp[-X]}{X} , \tag{4.39}$$

so that

$$\frac{\mathrm{d}^2 u}{\mathrm{d}X^2} - \left[A^2 - B \frac{\exp[-X]}{X} + \frac{Q(Q+1)}{X^2} \right] u = 0 , \tag{4.40}$$

where

$$A = 2\left(\frac{\delta\varkappa}{\varkappa}\right) , \qquad B = \frac{8}{\pi}\left(\frac{f^2}{4\pi}\right)\left(\frac{M}{\varkappa}\right)^2 . \tag{4.41}$$

The equation (4.40) is of the same form as that of the deuteron equation for a Yukawa potential. Such an equation for the deuteron has been studied extensively by HULTHÉN and LAURIKAINEN ([15]), using the variational method

([15]) L. HULTÉN and K. V. LAURIKAINEN: *Rev. Mod. Phys.*, **23**, 1 (1951); K. V. LAURIKAINEN: *Ann. Acad. Sci. Fenn.*, Ser. A, I, **130** (1952).

of Ritz and the method of the asymptotic expansions. In both cases the problem is treated in such a way that, when A^2 in (4.40) is taken as the known parameter, B is determined in terms of A^2 and Q. In the following we shall develop another method in which, when the parameter B is given, the eigenvalue A^2 is determined as a function of B and quantum numbers. The expression thus obtained is much more convenient in order to discuss the mass levels in terms of the coupling constant and quantum numbers.

First of all we note that the parameter B may be larger than unity for the actual problem, although the coupling constant itself may be smaller than unity. This comes from the fact that the factor $(8/\pi)(M/\varkappa)^2$ in (4.41) is of the order of magnitude of 100 at least for the case of the π-meson. Next, if we put $\exp[-X]$ in (4.40) equal to unity, by analogy with the energy levels of the hydrogen atom we find the following expression

$$A = \frac{B}{2(n+Q)}, \qquad (n = 1, 2, 3, \ldots), \tag{4.42}$$

where n is the *radial* quantum number for the wave function $u(X)$. It will be found that a straightforward application of the series expansion in powers of $(1/B)$, starting with (4.42) as a first term of the expansion, leads to approximate expressions for the mass levels for large values of B.

Now, writing

$$u(X) = \exp[-AX]w(X), \tag{4.43}$$

one obtains, from (4.40),

$$\frac{\mathrm{d}^2 w}{\mathrm{d}X^2} - 2A\frac{\mathrm{d}w}{\mathrm{d}X} + \left[B\frac{\exp[-X]}{X} - \frac{Q(Q+1)}{X^2}\right] w = 0 . \tag{4.44}$$

In order to consider the asymptotic equation for Eq. (4.44), we make the following assumptions:

1) The function w is a slowly varying function for large value of X. This means that we can ignore the term $(\mathrm{d}^2 w/\mathrm{d}X^2)$ compared to the term $-2A(\mathrm{d}W/\mathrm{d}X)$ for large X.

2) The eigenvalue A is a function of B. There exists the following finite limit C for $B \to \infty$:

$$\lim_{B\to\infty}\left(\frac{B}{2A}\right) = C , \qquad (C \neq 0). \tag{4.45}$$

Under these conditions, Eq. (4.44) reduces to the asymptotic equation

$$\frac{d\overline{w}}{dX} = C\frac{\exp[-X]}{X}\overline{w}\,, \qquad (B \to \infty). \tag{4.46}$$

3) All solutions of Eq. (4.44) tend to those of Eq. (4.46) in the limit of $B \to \infty$.

Now we postulate a solution of (4.46) in the form of a power series expansion in X,

$$\overline{w} = \sum_{\nu=0}^{\infty} \beta_\nu X^{\varrho'+\nu}\,, \qquad \beta_0 \neq 0,\ \varrho' \geqslant 0. \tag{4.47}$$

Inserting (4.47) into (4.46) and expanding $\exp[-X]$ into a power series one obtains the recurrence formula

$$\beta_\nu(\varrho'+\nu) = C\sum_{\sigma=0}^{\nu}\frac{(-1)^\sigma}{\sigma!}\beta_{\nu-\sigma}\,. \tag{4.48}$$

In order to determine ϱ', we assume that $\nu = 0$ and $\beta_0 = 1$. Then we find

$$\varrho' = C\,. \tag{4.49}$$

Accordingly, from (4.48) we get

$$\beta_\nu = \frac{C}{\nu}\sum_{\sigma=1}^{\nu}\frac{(-1)^\sigma}{\sigma!}\beta_{\nu-\sigma}\,. \tag{4.50}$$

In the same way, we assume a solution of Eq. (4.44) in the form

$$w = \sum_{\nu=0}^{\infty}\alpha_\nu X^{\varrho+\nu}\,, \qquad \alpha_0 \neq 0,\ \varrho \geqslant 0. \tag{4.51}$$

Substitution of (4.51) into (4.44) gives rise to the recurrence formula

$$[(\varrho+\nu+1)(\varrho+\nu) - Q(Q+1)]\alpha_{\nu+1} - 2A(\varrho+\nu)\alpha_\nu + B\sum_{\sigma=0}^{\nu}\frac{(-1)^\sigma}{\sigma!}\alpha_{\nu-\sigma} = 0\,. \tag{4.52}$$

Putting $\nu = -1$, we get the indicial equation

$$\varrho(\varrho-1) - Q(Q+1) = 0\,. \tag{4.53}$$

The solutions of (4.53) are

$$\varrho = Q + 1 , \qquad \varrho = -Q . \tag{4.54}$$

The boundary conditions for the wave function u are given by

$$u(0) = u(\infty) = 0 , \tag{4.55}$$

so that we merely take the solution, $\varrho = Q + 1$.

Up to now we have not assumed any special quantum numbers for the solutions (4.47) and (4.51). Both the recurrence formulas (4.50) and (4.52) are valid for all values of the quantum number Q. The recurrence formula (4.50) is, however, not independent of the radial quantum number n, so that from now on it is necessary to discuss the individual cases for the separate values of n. In what follows we only investigate the cases for $n = 1$ and $n = 2$. The same method can be applied directly to the other excited states.

a) First state, $n = 1$. – From (4.42), for the case of the Coulomb self-potential one obtains the relation

$$2A = \frac{B}{Q + 1} = \frac{B}{C} . \tag{4.56}$$

Now we introduce a parameter

$$\xi = \frac{1}{B} , \qquad (\xi < 1). \tag{4.57}$$

In order to get the expression of the mass levels for the case of the Yukawa type self-potential, we assume that the Coulomb level (4.56) is modified by the exponential factor $\exp[-X]$ in (4.44) in such a way that the mass level is expressed in terms of the following power series expansion in ξ

$$2A = \frac{1}{C\xi} + \sum_{\mu=0}^{\infty} a_\mu \xi^\mu , \tag{4.58}$$

and, at the same time, the coefficients α_ν in (4.51) are expanded as

$$\alpha_\nu = \sum_{\lambda=0}^{\infty} \alpha_{\nu,\lambda} \xi^\lambda . \tag{4.59}$$

Putting (4.58) and (4.59) into (4.52), we have

$$\sum_{\lambda=0}^{\infty} \Big\{ (\varrho + \nu + 1)(\varrho + \nu)\alpha_{\nu+1,\lambda}\xi^\lambda - Q(Q+1)\alpha_{\nu+1,\lambda}\xi^\lambda - \varrho^{-1}(\varrho + \nu)\alpha_{\nu,\lambda}\xi^{\lambda-1} - (\varrho + \nu) \sum_{\mu=0}^{\lambda} a_\mu \alpha_{\nu,\lambda-\mu}\xi^\lambda + \sum_{\sigma=0}^{\nu} \frac{(-1)^\sigma}{\sigma!} \alpha_{\nu-\sigma,\lambda}\xi^{\lambda-1} \Big\} = 0 ,$$

where use is made of $C = Q + 1 = \varrho$. Then one obtains for the coefficients a_μ and $a_{\nu,\lambda}$ the recurrence formula

$$(4.60)\qquad [(\varrho+\nu+1)(\varrho+\nu)-Q(Q+1)]\alpha_{\nu+1,\lambda}+\sum_{\sigma=0}^{\nu}\frac{(-1)^\sigma}{\sigma!}\alpha_{\nu-\sigma,\lambda+1}- \\ -\varrho^{-1}(\varrho+\nu)\alpha_{\nu,\lambda+1}-(\varrho+\nu)\sum_{\mu=0}^{\lambda}a_\mu\alpha_{\nu,\lambda-\mu}=0\,.$$

As to the initial values, we assume

$$(4.61)\qquad \alpha_{0,0}=1\,,\qquad \alpha_{0,\lambda}=0\,,\qquad (\lambda\neq 0),$$

and according to the assumption 3)

$$(4.62)\qquad \alpha_{\nu,0}=\beta_\nu\,,\qquad \beta_0=1\,.$$

We are in a position to determine the coefficients successively. It is easily shown from the relation (4.50) that

$$(4.63)\qquad \begin{cases} \beta_0=\alpha_{0,0}=1\,,\\ \beta_1=\alpha_{1,0}=-C\,,\\ \beta_2=\alpha_{2,0}=(C/4)(2C+1)\,,\\ \beta_3=\alpha_{3,0}=-(C/36)(6C^2+9C+2)\,,\\ \beta_4=\alpha_{4,0}=(C/288)(12C^3+36C^2+25C+3)\,,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

The coefficients a_μ and $\alpha_{\nu,\lambda}$ are determined by (4.60) in the following way, Putting $\nu=0$ in (4.60), one finds

$$[(\varrho+1)\varrho-Q(Q+1)]\alpha_{1,\lambda}=\varrho a_\lambda\alpha_{0,0}\,,$$

so that

$$(4.64)\qquad a_\lambda=\left(\frac{2}{\varrho}\right)(Q+1)\alpha_{1,\lambda}=2\alpha_{1,\lambda}\,.$$

It is clear from the above relation that the power series of the mass levels is defined completely by the coefficients $\alpha_{1,\lambda}$.

First, let us consider a_0. If we take the values $\nu=0$ and $\lambda=0$ in (4.60), we get

$$-[(\varrho+1)\varrho-Q(Q+1)]\varrho-\varrho a_0=0\,.$$

This leads to

$$a_0 = -2(Q+1)\,, \tag{4.65}$$

by using $\varrho = Q+1$.

Next consider a_1. For the case when $\nu = 1$ and $\lambda = 0$, Eq. (4.60) becomes

$$\begin{cases} [(\varrho+2)(\varrho+1) - Q(Q+1)]\alpha_{2,0} + \alpha_{1,1} \\ -\varrho^{-1}(\varrho+1)\alpha_{1,1} - (\varrho+1)a_0\alpha_{1,0} = 0\,. \end{cases} \tag{4.66}$$

With the aid of the (4.63), Eq. (4.66) is reduced to

$$\alpha_{1,1} = \tfrac{1}{2}\varrho^2\,. \tag{4.67}$$

Hence

$$a_1 = \varrho^2\,. \tag{4.68}$$

Now, let us determine a_2. If we choose $\nu = 2$ and $\lambda = 0$ in (4.60), we find after some calculations

$$\alpha_{2,1} = -\tfrac{1}{6}\varrho^2(4\varrho+1)\,. \tag{4.69}$$

Putting $\nu = 1$ and $\lambda = 1$ in (4.60), and using (4.69), we have

$$\alpha_{1,2} = -\tfrac{1}{3}\varrho^3(2\varrho^2+1)\,. \tag{4.70}$$

Therefore, finally it follows from (4.64) that

$$a_2 = -\tfrac{2}{3}\varrho^3(2\varrho^2+1)\,. \tag{4.71}$$

In this way, one can get $a_\mu\xi^\mu$ to any order in ξ. We shall, however, be content to record the series up to the term $a_2\xi^2$. The expression (1.58) is now written as

$$A_{n=1} = \frac{B}{2(Q+1)} - (Q+1) + (Q+1)^2\left(\frac{1}{2B}\right) - (Q+1)^5[2(Q+1)^2+1]\left(\frac{1}{3B^2}\right) + \cdots\,, \tag{4.72}$$

where use is made of $\varrho = Q+1$. The first term on the right-hand side is the Coulomb term, and the other terms indicate the effect of $\exp[-X]$ in (4.44).

b) *Second state*, $n = 2$. – We now proceed to discuss the next excited states with respect to the quantum number n. For the case of $n = 2$, the Coulomb level is written as

$$2A = \frac{B}{Q+2} = \frac{B}{C}, \tag{4.73}$$

with the help of (4.42) and (4.45). Consequently, the expansion of the eigenvalue $2A$ with respect to ξ is of the form

$$2A = \frac{1}{(Q+2)\xi} + \sum_{\mu=0}^{\infty} b_\mu \xi^\mu . \tag{4.74}$$

Here we employ the same notation as in the case of $n = 1$ except for b_μ:

First of all the asymptotic equation of this case becomes

$$\frac{\mathrm{d}\overline{w}}{\mathrm{d}X} = (Q+2)\frac{\exp[-X]}{X}\overline{w}, \tag{4.75}$$

under the condition

$$\lim_{B\to\infty}\left(\frac{B}{2A}\right) = C = Q+2 . \tag{4.76}$$

The solution of Eq. (4.75) may be of the form

$$\overline{w} = \sum_{\nu=0}^{\infty} \beta_\nu W^{\varrho+2+\nu}, \tag{4.77}$$

by analogy with the solutions (4.47) and (4.49). It may be inferred from the expansion (4.51) that, for the present case, Eq. (4.44) has the solution

$$w = \sum_{\nu=0}^{\infty} \alpha_\nu X^{\varrho+\nu} = \sum_{\nu=0}^{\infty} \alpha_\nu X^{\varrho+1+\nu}, \tag{4.78}$$

where α_ν are expanded in the form

$$\alpha_\nu = \sum_{\lambda=0}^{\infty} \alpha_{\nu,\lambda}\xi^\lambda , \tag{4.79}$$

and we have used the relation $\varrho = Q + 1$ arising from the indicial equation. However, the coefficients $\alpha_{\nu,\lambda}$ are different from those for the case of $n = 1$ in that they are required to have another correspondence to β_ν and new initial values. First, according to the condition 3), we have the relation

$$\alpha_{\nu+1} \to \beta_\nu , \qquad (B \to \infty) . \tag{4.80}$$

Consequently, we assume

$$\alpha_{\nu+1,0} = \beta_\nu \,, \tag{4.81}$$

instead of (4.62). Next, one takes into account the special condition imposed on the coefficient α_0. As is easily seen from (4.77) and (4.78), we have a demand that α_0 becomes zero for $B \to \infty$ in accordance with the condition that $w \to \overline{w}$ in the limit of $B \to \infty$. This can be accomplished by supposing that α_0 is of the form

$$\alpha_0 = \frac{1}{B} = \xi \,. \tag{4.82}$$

Finally, we choose the initial values corresponding to (4.61) as

$$\alpha_{0,1} = 1 \,, \qquad \alpha_{0,\lambda} = 0 \,, \qquad (\lambda \neq 1). \tag{4.83}$$

For β_ν we have the recurrence formula

$$\beta_\nu = \frac{(\varrho+1)}{\nu} \sum_{\sigma=1}^{\nu} \frac{(-1)^\sigma}{\sigma!} \beta_{\nu-\sigma} \,. \tag{4.84}$$

The recurrence formula for $\alpha_{\nu,\lambda}$ is found to be

$$[(\varrho+\nu+1)(\varrho+\nu) - Q(Q+1)]\alpha_{\nu+1,\lambda} + \sum_{\sigma=0}^{\nu} \frac{(-1)^\sigma}{\sigma!} \alpha_{\nu-\sigma,\lambda+1} - \frac{(\varrho+\nu)}{(\varrho+1)} \alpha_{\nu,\lambda+1} - (\varrho+\nu) \sum_{\mu=0}^{\lambda} b_\mu \alpha_{\nu,\lambda-\mu} = 0 \,. \tag{4.85}$$

As in the case of $n = 1$, we can determine β_ν as follows:

$$\begin{cases} \beta_0 = \alpha_{1,0} = \dfrac{1}{2\varrho(\varrho+1)} \,, \\ \beta_1 = \alpha_{2,0} = \dfrac{1}{2\varrho} \,, \\ \beta_2 = \alpha_{3,0} = -\left(\dfrac{1}{8\varrho}\right)(2\varrho+3) \,, \\ \beta_3 = \alpha_{4,0} = \left(\dfrac{1}{72\varrho}\right)(6\varrho^2 + 21\varrho + 17) \,, \\ \beta_4 = \alpha_{5,0} = -\left(\dfrac{1}{576\varrho}\right)(12\varrho^3 + 72\varrho^2 + 133\varrho + 76) \,, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{4.86}$$

We write down the expressions for $\alpha_{\nu,\lambda}$ which are obtained from the formula (4.85) by choosing the pairs of numerical values for the suffixes ν and λ (we denote them by (ν, λ)) as follows:

$$(4.87) \qquad (0, \lambda): \qquad b_\lambda = 2\alpha_{1,\lambda+1},$$

$$(4.88) \qquad (1, 0): \qquad b_0 = 2\alpha_{1,1} = -2(\varrho+1),$$

$$(4.89) \qquad (2, 0): \qquad \alpha_{2,1} = (1/4\varrho)(\varrho+1)(2\varrho^2+3\varrho-1),$$

$$(4.90) \qquad (3, 0): \qquad \alpha_{3,1} = -(1/36\varrho)(\varrho+1)(6\varrho^3+21\varrho^2+5\varrho-15),$$

$$(4.91) \qquad (1, 1): \qquad b_1 = 2\alpha_{1,2} = (\varrho+1)(3\varrho+1),$$

$$(4.92) \qquad (2, 1): \qquad \alpha_{2,2} = (1/6\varrho)(\varrho+1)^2(-8\varrho^2-5\varrho+3),$$

$$(4.93) \qquad (1, 2): \qquad b_2 = 2\alpha_{1,3} = -(2/3)(\varrho+1)^3\{2(\varrho+1)^2+1\},$$

$$\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots$$

With the help of these expressions, we get finally from (4.74) the power series expansion of the mass eigenvalues,

$$(4.94) \qquad A_{n=2} = \frac{B}{2(Q+2)} - (Q+2) + (Q+2)[2(Q+2)+Q]\left(\frac{1}{2B}\right) - \\ - (Q+2)^3[2(Q+2)^2+1]\left(\frac{1}{3B^2}\right) + \ldots .$$

In general, this method can be applied to any other states. Of course, the expansions discussed above are not completely satisfactory from the point of view of convergence. For large values of B, the expansions are, however, suitable for a rough estimation of the eigenvalue of mass and for an understanding of the physical significance of the excited states which seem to correspond to heavy mesons.

We now proceed to discuss the mass levels in connection with the heavy mesons. So far we have retained only the first term of the self-potential. It must therefore be expected that the mass levels obtained above have only a qualitative significance. The following discussion is intended as a substitute for a more complete future theory of the mass levels. As for the coupling constants, it seems doubtful to discuss the mass levels merely in terms of the coupling constant f in view of the various possible interactions between the mesons and nucleons. It seems, however, worth while at this stage to restrict

ourselves to such an interaction and to investigate the results obtained by the model. It is our primary concern whether the theory based on the usual pseudoscalar and pseudovector couplings leads to reasonable mass levels, which are not incompatible with the results indicated by recent experiments, at least in the order of the magnitudes of masses.

In order to estimate the mass values of the mesons, we first evaluate A in terms of B by making use of the expressions (4.72) and (4.94). The numerical values thus obtained are tabulated in Table I. These values are calculated

TABLE I. – *The values of A for the ground, first excited and second excited states.*

B	A $n=1,\ Q=\frac{1}{2}$	A $n=2,\ Q=\frac{1}{2}$	A $n=1,\ Q=\frac{3}{2}$
4	− 0.272	—	—
6	+ 0.516	—	—
8	1.211	—	—
10	1.884	—	—
12	2.551	− 0.015	− 0.328
14	3.216	+ 0.432	+ 0.165
16	3.880	0.855	0.621
18	4.543	1.265	1.057
20	5.207	1.668	1.481
22	5.872	2.067	1.897
24	6.537	2.464	2.308
26	7.202	2.866	2.719
28	7.866	3.256	3.122
30	8.351	3.651	3.526
32	9.196	4.046	3.929
34	9.861	4.441	4.331
36	10.526	4.837	4.733
38	11.192	5.232	5.134
40	11.858	5.628	5.534

for the three different states of the meson: the ground state $n=1$, $Q=\frac{1}{2}$, the first excited state $n=2$, $Q=\frac{1}{2}$, and the second excited state $n=1$, $Q=\frac{3}{2}$. Here we remark that the Table I can be used in other cases when the expressions for A and B are different from those of (4.41), so long as the radial wave equations are of the form (4.40).

Next, we discuss the masses of the mesons. Let us recall the basic assumption concerning the actual mass $\varkappa$:

$$\varkappa = \varkappa_0 - \delta\varkappa\,, \qquad (\delta\varkappa > 0), \tag{4.95}$$

where $\varkappa_0$ is a bare mass which is assumed to be *positive* and finite. We cannot determine $\varkappa$ without knowing the exact value of $\varkappa_0$. However, we can determine the value of $\varkappa$ as well as that of $\varkappa_0$ by the following consideration. From the experimental results, one may be permitted to assume that the π-meson corresponds to a ground state in the so-called π-meson family. Therefore we assign the ground state ($n = 1$, $Q = \frac{1}{2}$) to the π-meson. In order to make our discussion clear, we choose a numerical value of B in the Table I, namely

$$B = 32 . \tag{4.96}$$

Then, from (4.41) we find that the coupling constant is given by

$$\frac{f^2}{4\pi} = \left(\frac{\pi}{8}\right)\left(\frac{\varkappa}{M}\right)^2 B = 0.278 , \tag{4.97}$$

where we assume that $M = 1836\, m_e$ and $\varkappa = 273\, m_e$. From the Table I and (4.41), the expression of A corresponding to (4.96) is found to be

$$A = \frac{2\delta\varkappa}{\varkappa} = 9.196 , \tag{4.98}$$

so that

$$\frac{\delta\varkappa}{\varkappa} = 4.598 . \tag{4.99}$$

With the help of (4.95) and (4.99), one obtains

$$\frac{\varkappa_0}{\varkappa} = 1 + \frac{\delta\varkappa}{\varkappa} = 5.598 , \quad \text{i.e.} \quad \varkappa_0 = 1528.2\, m_e . \tag{4.100}$$

For the first excited state ($n = 2$, $Q = \frac{1}{2}$), from the Table I we get

$$A = \frac{2\delta\varkappa_1}{\varkappa} = 4.046 , \quad \text{i.e.} \quad \frac{\delta\varkappa_1}{\varkappa} = 2.023 . \tag{4.101}$$

From (4.95) and (4.101), it follows that the first excited state has the mass

$$\varkappa_1 = \varkappa_0 - \delta\varkappa_1 = 975.9\, m_e . \tag{4.102}$$

In the same way, for the second excited state ($n = 1$, $Q = \frac{3}{2}$) we find

$$\varkappa_2 = 991.9\, m_e . \tag{4.103}$$

In Table II, the coupling constants, the bare masses, and the masses of the heavy mesons, which are obtained for various values of B are given. It is easily seen that the mass values for the excited states are of the same order as those of the observed heavy mesons (~ 970 m_e). It should be noted that for the values of B smaller than 14 there is no excited state. Even for relatively large values of B, we merely have a finite number of excited states, which is a remarkable feature of the so-called Yukawa-type potential.

TABLE II. – *The numerical values of the bare mass* $\varkappa_0$, *the first excited state mass* $\varkappa_1$ *and the second excited state mass* $\varkappa_2$ *in units of* m_e. $M = 1836\ m_e$; $\varkappa_\pi = 273\ m_e$ $(n = 1,\ Q = \frac{1}{2})$.

B	$f^2/4\pi$	$\varkappa_0$	$\varkappa_1$ $(n=2,\ Q=\frac{1}{2})$	$\varkappa_2$ $(n=1,\ Q=\frac{3}{2})$
14	0.122	711.9 m_e	652.9 m_e	689.5 m_e
16	0.139	802.5	685.8	717.8
18	0.156	893.2	720.5	748.9
20	0.174	983.6	756.1	781.7
22	0.191	1074.5	791.8	815.6
24	0.208	1165.2	828.8	850.1
26	0.226	1256.0	864.9	884.9
28	0.243	1346.7	902.2	920.5
30	0.260	1437.4	939.0	956.1
32	0.278	1528.2	975.9	991.9
34	0.295	1619.0	1012.8	1027.8
36	0.313	1709.9	1049.7	1063.9
38	0.330	1800.7	1086.5	1097.9
40	0.347	1891.5	1123.3	1136.1

The situation is, however, slightly different if we consider the other case of the partition of $\varkappa$, namely, $\varkappa$ consists of two parts, $\varkappa_0$ and $-\delta\varkappa$, in the (x'_μ, x''_μ) spaces. We shall summarize here the result obtained for this case. The corresponding expressions for A and B are

$$A = \frac{1}{\varkappa_0}\left[(\varkappa_0 + \delta\varkappa)^2 + \delta\varkappa^2\right]^{\frac{1}{2}}, \qquad (\varkappa = \varkappa_0 - \delta\varkappa), \tag{4.104}$$

and

$$B = \frac{2}{\pi}\left(\frac{f^2}{4\pi}\right)\left(\frac{M}{\varkappa_0}\right)^2, \tag{4.105}$$

respectively. Eq. (4.104) is written as

$$\left(\frac{\varkappa}{\varkappa_0}\right)^2 - 3\left(\frac{\varkappa}{\varkappa_0}\right) + \frac{1}{2}(5 - A^2) = 0\,. \tag{4.106}$$

We notice that the observed mass $\varkappa$ is always smaller than the bare mass $\varkappa_0$. Then it follows from (4.106) that

$$\left(\frac{\varkappa}{\varkappa_0}\right) = \tfrac{1}{2}(3 \pm \sqrt{2A^2 - 1}) \leqslant 1 , \tag{4.107}$$

which implies

$$1 \leqslant A \leqslant 2.236 . \tag{4.108}$$

Consequently, it turns out from the Table I that there is no excited state if we suppose that the ground state corresponds to the π-meson. This means that we cannot determine the bare mass without arbitrariness. Therefore, we shall not go any further into the analysis of this case.

We have so far discussed the method of how to estimate the mass eigenvalues for the simple potential form $V_1(X)$. Although $V_1(X)$ is a part of the self-potential $V(X)$ in the second-order approximation, it is useful in deciding qualitatively the general feature of the mass levels partly because of its simple analytical form, and partly because of its higher singularity than the other parts. We thus gain an insight into the intimate relation between the so-called divergence and the quantized masses of the particles. As is well known, the difficulties of the divergences is the main problem which mars the success of present quantum field theory. It is, however, possible, by quantum mechanical analogues, to overcome some of the difficulties and, at the same time, to throw light on the problem of mass-quantization.

We conclude this section with the remark that the estimation of the mass levels was carried out merely to illustrate the method developed above and not because of the belief in the model being in accordance with the actual cases of the heavy mesons (*). This problem will be discussed in a future paper.

5. – Variational Method.

In this Section we shall investigate our problem by making use of the variational method developed by LAURIKAINEN and EURANTO ([16]) for the case of the deuteron problem. The main purpose of this Section is to estimate the effect of the self-potential other than $V_1(X)$ discussed in the previous Section.

According to (3.115) and (3.116), the divergent parts of the self-potential

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(*) The first excited state may correspond to the so-called τ-meson.

([16]) K. V. LAURIKAINEN and E. K. EURANTO: *Turun Yliopiston Julkaisuja Ann. Univ. Turkuensis*, Ser. A, **9**, No. 3 (1953).

are written as

$$V_1(X) = \frac{1}{2\pi^2} f^2 M^2 \frac{\exp[-X]}{X}, \tag{5.1}$$

and

$$V_2(X) = \frac{1}{2\pi^2} f^2 M^2 \left\{1 - \frac{1}{2}\left(\frac{\varkappa}{M}\right)^2\right\} E_i(-X). \tag{5.2}$$

For simplicity, we assume that the finite term of the self-potential is discarded and $\varkappa < M$. Then the self-potential is given by

$$V(X) = V_1(X) + V_2(X) > 0, \quad V_1(X) > 0, \quad V_2(X) < 0. \tag{5.3}$$

In order to compare $V_2(X)$ with $V_1(X)$, we give here some numerical values $\exp[-X]/X$ and $-Ei(-X)$ for several values of X:

X	0.1	0.2	0.5	1	2	5
$\exp[-X]/X$	9.0	4.0	1.2	0.4	0.07	0.001
$-Ei(-X)$	1.8	1.2	0.6	0.2	0.05	0.001

The correction of $V_2(X)$ to $V_1(X)$ depends on the values of X. For large values of X, the correction part becomes comparable with $V_1(X)$. However, for small X, the contribution from $V_2(X)$ is not important compared to $V_1(X)$. The sign of $V_2(X)$ is different from that of $V_1(X)$, so that the effect of $V_2(X)$ is considered as a screening effect to $V_1(X)$.

Now we take into account the effect due to $V_2(X)$ through the following replacement in the expression of $V_1(X)$,

$$f^2 \to (1 - \varepsilon) f^2, \tag{5.4}$$

where ε indicates a numerical factor being smaller than unity. This factor is determined in the following way. First, we apply the variational method to the case when we retain only $V_1(X)$. It is convenient to consider, in this case, that the eigenvalues to be determined are the coupling constants while the mass values are supposed to be known parameters. The reason for this choice is that the masses are involved in the wave functions to be decided by the variational method, in a complicated manner. Then we can get the wave functions which minimize the eigenvalues of the coupling constants. Next, with the help of the wave functions obtained, we calculate the ratio described by

$$\left| \frac{(u^*(X) V_2(X)\, u(X))}{(u^*(X) V_1(X)\, u(X))} \right| = \varepsilon. \tag{5.5}$$

The factor ε thus gained is inserted into the expression of the replacement (5.4).

This procedure is therefore identical with the introduction of the effective potential from the beginning:

$$V_{\text{eff}}(X) = \frac{1}{2\pi^2} f^2 M^2 (1-\varepsilon) \frac{\exp[-X]}{X}, \tag{5.6}$$

nstead of $V(X)$ given by (5.3). Thus we consistently ignore the contribution of $V_2(X)$ to $V_1(X)$, assuming that $V_2(X)$ has been properly taken into account by the numerical factor ε. The explicit calculations of (5.5) show that ε is $\sim 0.5-0.4$. Rigorously, one can replace the averaging process mentioned above by a more exact procedure based on the perturbation method. However, in a qualitative sense, the ratio ε reflects the effect of $V_2(X)$ in such a way that the result is not considerably different from that for the exact method.

Making use of ε, we can easily infer the mass values as follows. For the effective self-potential $V_{\text{eff}}(X)$, we can replace the expression (4.41) by

$$B = (1-\varepsilon)\left(\frac{8}{\pi}\right)\left(\frac{f^2}{4\pi}\right)\left(\frac{M}{\varkappa}\right)^2. \tag{5.7}$$

Instead of considering the original B, we use the expression (5.7) and discuss the mass levels in terms of the Tables I and II as in the case of the previous section. Thus the estimate of the effect of $V_2(X)$ on $V_1(X)$ can be accomplished merely by changing the coupling constant according to (5.4).

The exact treatment of mass levels for the complete potential $V(X)$ may be in general tedious. On the other hand, although the approximation employed here is somewhat crude, the present method furnishes easily the qualitative features of the mass levels for $V(X)$.

As is well known, the wave equation (4.40),

$$\frac{\mathrm{d}^2 u}{\mathrm{d}X^2} - \left[A^2 - B\frac{\exp[-X]}{X} + \frac{Q(Q+1)}{X^2}\right] u = 0,$$

is transformed into the following variational problem for the quantity D:

$$D = \frac{J}{N}, \tag{5.8}$$

where

$$J = \int_0^\infty \left[\left(\frac{\mathrm{d}u}{\mathrm{d}X}\right)^2 + A^2 u^2 + \frac{Q(Q+1)}{X^2} u^2\right] \mathrm{d}X, \tag{5.9}$$

$$N = \int_0^\infty \frac{\exp[-X]}{X} u^2 \, \mathrm{d}X, \qquad (Q = \tfrac{1}{2}, \tfrac{3}{2}, \ldots). \tag{5.10}$$

and A is assumed to be a known parameter. We suppose that the unnormalized wave function is of the form

$$u(X) = \exp[-AX](1-\exp[-X])^{Q+1}\sum_{j=0}^{k} h_j \exp[-jX]\,. \tag{5.11}$$

This is a suitable trial function in which h_j are unknown parameters. For large value of X, $u(X)$ falls off rapidly, and, for $X \to 0$, $u(X)$ approaches zero in the same way as X^{Q+1}. These properties of the wave function $u(X)$ are required by the consideration in the preceding section. In what follows the calculation concerning the case $k = 2$ will be discussed in detail. Moreover, we confine ourselves to the states having the quantum number $Q = \frac{1}{2}$. This means that we merely consider the ground state and the first excited state.

Inserting (5.11) into (5.9) and (5.10), we have

$$\begin{aligned} J = {} & h_0^2(L_0 - M_0 + P_0) + 2h_0h_1(L_1 - M_1 + P_1) + \\ & + (h_1^2 + 2h_0h_2)(L_2 - M_2 + P_2) + h_1^2 I_2 + \\ & + 2h_1h_2(L_3 - M_3 + P_3 + 2I_3) + h_2^2(L_4 - M_4 + P_4 + 2I_4)\,, \end{aligned} \tag{5.12}$$

$$N = h_0^2 N_0 + 2h_0h_1N_1 + (h_1^2 + 2h_0h_2)N_2 + 2h_1h_2N_3 + h_2^2 N_4\,; \tag{5.13}$$

where

$$L_\mu = \left(3A + \frac{3}{2}\right)\int_0^\infty \exp[-(2A+1+\mu)X](1-\exp[-X])^2\,\mathrm{d}X\,, \tag{5.14}$$

$$M_\mu = \frac{3}{4}\int_0^\infty \exp[-(2A+2+\mu)X](1-\exp[-X])\,\mathrm{d}X\,, \tag{5.15}$$

$$P_\mu = \frac{3}{4}\int_0^\infty \exp[-(2A+\mu)X](1-\exp[-X])^3\left(\frac{1}{X}\right)^2\mathrm{d}X\,, \tag{5.16}$$

$$I_\nu = \int_0^\infty \exp[-(2A+\nu)X](1-\exp[-X])^3\,\mathrm{d}X\,, \tag{5.17}$$

$$N_\mu = \int_0^\infty \exp[-(2A+1+\mu)X](1-\exp[-X])^3\left(\frac{1}{X}\right)\mathrm{d}X\,, \tag{5.18}$$

$$(\mu = 0, 1, 2, 3, 4), \quad (\nu = 2, 3, 4)\,.$$

In evaluating the integrals over X, we shall use the following formulas. Let us consider Euler's integral of first kind ([17])

$$(5.19)\qquad B(p, q+1) = \int_0^\infty \exp[-py](1-\exp[-y])^q \,dy = \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}.$$

If q is a positive integer, we find

$$(5.20)\qquad B(p, q+1) = \frac{1\cdot 2\cdot 3\cdots q}{p(p+1)\cdots(p+q)}.$$

Accordingly, using this formula, we get the expressions for L_μ, M_μ and I_ν,

$$(5.21)\qquad L_\mu = 2(3A+\tfrac{3}{2})[(2A+1+\mu)(2A+2+\mu)(2A+3+\mu)]^{-1},$$

$$(5.22)\qquad M_\mu = (\tfrac{3}{4})[(2A+2+\mu)(2A+3+\mu)]^{-1},$$

and

$$(5.23)\qquad I_\nu = 6[(2A+\nu)(2A+1+\nu)(2A+2+\nu)(2A+3+\nu)]^{-1}.$$

Another formula which is useful in evaluating N_μ is

$$\int_0^\infty \exp[-cX](1-\exp[-X])^3\left(\frac{1}{X}\right)dX = \log\left(\frac{3+c}{c}\right) + 3\log\left(\frac{1+c}{2+c}\right),$$

where use is made of the relation

$$\int_0^\infty (\exp[-bX]-\exp[-aX])\left(\frac{1}{X}\right)dX = \log\left(\frac{a}{b}\right).$$

With this formula, N_μ can be written as

$$(5.24)\qquad N_\mu = \log\left(\frac{2A+4+\mu}{2A+1+\mu}\right) + 3\log\left(\frac{2A+2+\mu}{2A+3+\mu}\right).$$

In the same way, P_μ is found to be

$$(5.25)\qquad P_\mu = (\tfrac{3}{4})[(2A+\mu)\log(2A+\mu) - 3(2A+1+\mu)\log(2A+1+\mu) + \\ + 3(2A+2+\mu)\log(2A+2+\mu) - (2A+3+\mu)\log(2A+3+\mu)].$$

([17]) E. T. Whittaker and G. N. Watson: *Modern Analysis* (Cambridge, 1920), Third Edition, p. 253.

The conditions for the extremum of the expression (5.8) are readily found

$$\frac{\partial J}{\partial h_j} - \lambda \frac{\partial N}{\partial h_j} = 0 , \qquad (j = 0, 1, 2), \tag{5.26}$$

where λ are the eigenvalues of D. Eqs. (5.26) lead to simultaneous linear equations with respect to h_j. The eigenvalues λ are decided by the determinant equation of these equations. The real and positive values of λ give us approximate eigenvalues of B corresponding to the first two low states. After a simple manipulation we obtain the determinant equation

$$\begin{vmatrix} L_0 - M_0 - \lambda N_0 + P_0 , & L_1 - M_1 - \lambda N_1 + P_1 , & L_2 - M_2 - \lambda N_2 + P_2 \\ L_1 - M_1 - \lambda N_1 + P_1 , & L_2 - M_2 - \lambda N_2 + P_2 + I_2 , & L_3 - M_3 - \lambda N_3 + P_3 + 2I_3 \\ L_2 - M_2 - \lambda N_2 + P_2 , & L_3 - M_3 - \lambda N_3 + P_3 + 2I_3 , & L_4 - M_4 - \lambda N_4 + P_4 + 2I_4 \end{vmatrix} = 0, \tag{5.27}$$

Hitherto attention has been concentrated exclusively on the general expression of the variational method. We shall now first consider the case where the parameter A has a special value 0.5. Then, Eqs. (5.21)–(5.27) combine to give

$$\lambda^3 - 5.594\lambda^2 - 219.559\lambda + 1269.562 = 0 . \tag{5.28}$$

The solutions of this equation are

$$\lambda_1 = 5.817 , \qquad \lambda_2 = 14.662 . \tag{5.29}$$

Here we ignore the third, negative real solution. For the case of λ_1, let us evaluate the factor ε defined by (5.5). One of the corresponding sets of h_j is obtained from (5.26) as follows:

$$h_0 = 1 , \qquad h_1 = 1.546 , \qquad h_2 = 0.025 , \tag{5.30}$$

where, for brevity, h_0 is taken to be unity. Therefore, from (5.11) one gets the wave function

$$u(X) = \exp[-0.5X](1 - \exp[-X])^{1.5}(h_0 + h_1 \exp[-X] + h_2 \exp[-2X]) , \tag{5.31}$$

where h_j are given by (5.30) and the wave function is not normalized. According to (5.1), the expectation value of $V_1(X)$ becomes

$$(u^*(X) V_1(X) u(X)) = \int_0^\infty \overline{8u^* u} \frac{\exp[-X]}{X} \, \mathrm{d}X . \tag{5.32}$$

Here we put

$$S = \frac{1}{2\pi^2} f^2 M^2 . \tag{5.33}$$

Employing the wave function (5.31), we find

$$(u^* V_1 u) = 0.158\, S . \tag{5.34}$$

The expectation value of $V_2(X)$ is of the form

$$(u^* V_2 u) = -S \left\{ 1 - \frac{1}{2} \left(\frac{\varkappa}{M} \right)^2 \right\} \int_0^\infty u^* u [- E_i(-X)] \, \mathrm{d}X . \tag{5.35}$$

The integration cannot be carried out analytically. We invoke the method of numerical calculation. Let $F(X)$ be the integrand of (5.35),

$$F(X) = \exp[-X](1 - \exp[-X])^3 \cdot \cdot (h_0 + h_1 \exp[-X] + h_2 \exp[-2X])^2 [- E_i(-X)] . \tag{5.36}$$

It is easily seen that

$$\left. \frac{\mathrm{d}F(X)}{\mathrm{d}X} \right|_{X \to 0} = 0 , \qquad \left. \frac{\mathrm{d}F(X)}{\mathrm{d}X} \right|_{X \to \infty} = 0 . \tag{5.37}$$

Let us divide the interval ($X_0 = 0$, $X_n = \infty$) into n ($n \to \infty$) equal parts whose lengths are δX. If the derivatives of the integrand vanish at the end points X_0 and X_n, we have the formula ([18])

$$\int_{X_0}^{X_n} F(X) \, \mathrm{d}X = \tag{5.38}$$

$$= \tfrac{1}{2}(\delta X)[F(X_0) + 2F(X_1) + 2F(X_2) + \ldots + 2F(X_{n-1}) + F(X_n)] .$$

For practical calculations, it is, however, sufficient to retain at most scores of terms on the right-hand side of (5.38). For example, for $X = 1$, we see that $F(1) = 0.0504$. For $X = 3$, on the other hand, we find that $F(3) = 0.0006$. The further terms on the right-hand side decrease rapidly as X_n become large. Using these expressions, for the assumed value $\delta X = 0.2$, the expectation value of $V_2(X)$ is found to be

$$(u^* V_2 u) = -0.079 S \left\{ 1 - \frac{1}{2} \left(\frac{\varkappa}{M} \right)^2 \right\} . \tag{5.39}$$

([18]) D. R. HARTREE: *Numerical Analysis* (Oxford, 1952), p. 101.

If the π-meson is assigned to the ground state, it follows that

$$\left\{1-\frac{1}{2}\left(\frac{\varkappa}{M}\right)^2\right\} = 0.989\,, \quad (M = 1\,836\ \mathrm{m_e},\ \varkappa = 273\ \mathrm{m_e}).$$

Finally, fom the relations (5.34) and (5.39) we obtain

$$\varepsilon = 0.498\,. \tag{5.40}$$

Therefore, the effect of $V_2(X)$ amounts to nearly half of that due to $V_1(X)$ for the values $A = 0.5$, $n = 1$ and $Q = 0.5$.

The same method can be applied to other cases. Here we write down some results obtained for the cases of $A = 1$ and $A = 2$.

i) $A = 1$, $Q = 0.5$:

$$\lambda_1 = 7.271\,, \qquad \lambda_2 = 18.713\,. \tag{5.41}$$

For λ_1,

$$h_0 = 1, \qquad h_1 = 1.192\,, \qquad h_2 = 0.376\,, \tag{5.42}$$

$$\varepsilon = 0.451\,. \tag{5.43}$$

ii) $A = 2$, $Q = 0.5$:

$$\lambda_1 = 10.295\,, \qquad \lambda_2 = 22.976\,. \tag{5.44}$$

For λ_1,

$$h_0 = 1\,, \qquad h_1 = 1.270\,, \qquad h_2 = 0.254\,, \tag{5.45}$$

$$\varepsilon = 0.391\,. \tag{5.46}$$

Thus we find that the values of the coupling constants for the total self-potential $V(X)$ differ from the values for the Yukawa type self-potential $V_1(X)$ by an amount of the order of $0.5 - 0.4$ for the coupling constants less than 0.2. By taking account of these corrections, we see that the Tables I and II will still be useful for the estimation of the mass levels of the mesons.

6. – Concluding Remarks.

The main feature of the theory developed in this paper is its success in avoiding the divergence difficulty of the self-mass of the meson, and at the same time in furnishing mass spectra which seem to be qualitatively not at

variance with experimental results, at least in the lowest order approximation. The method here given for the treatment of the self-mass of an elementary particle is quite general. It is hoped that this method can be applied to the hyperon mass spectra ([19]) leading to reasonable results.

It may be stressed, on the other hand, that the theory itself is not completely free from ambiguities which are encountered in the cases of the decomposition of the observed mass of the meson into fictitious parts, and of the assumption on the mutual relation between the proper times of particles. It is, however, obvious that the basic equations such as (2.15) and (2.16) have nothing to do with the ambiguities. Thus these equations appear to be of much wider generality. We shall pursue the ambiguities further elsewhere. We believe that the main physical consequences of the use of the relativistic invariant wave equations in the case of mass-quantization have been exposed by the present approach.

We should like to point out another important feature of the theory. So far, nothing has been done other than the usual treatment for the divergences in the external space. Indeed, we have made use of the theory of renormalization, which seems to be applicable to the cases of the pseudovector coupling as well as the pseudoscalar coupling ([20]), and which is necessary also to secure the causality condition in the external space ([21]). We have merely grafted the internal space upon the external space. Accordingly, what we expect in a correct future theory is not a modification of the conventional theories which give us excellent agreement with experimental results in quantum electrodynamics, but rather an expansion of the theory, so that it can describe the internal structure of elementary particels.

* * *

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([19]) The models of the hyperons are discussed in our paper: H. ENATSU and C. IHARA: *Nuovo Cimento*, **1**, 394 (1955).

([20]) R. ARNOWITT and S. DESER: *Phys. Rev.*, **100**, 349 (1955); L. H. COOPER: *Phys. Rev.*, **100**, 362 (1955).

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